

6. DETERMINANTS

Determinant of a Square Matrix

Corresponding to each square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix},$$

there is associated an expression, called the *determinant* of A , denoted by $\det A$, or $|A|$, written as

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}.$$

A matrix is an arrangement of numbers and so it has no fixed value, while each determinant has a fixed value.

A determinant having n rows and n columns is known as a determinant of order n .

The determinants of nonsquare matrices are not defined.

VALUE OF A DETERMINANT OF ORDER 1 *The value of a determinant of a (1×1) matrix $[a]$ is defined as $|a| = a$.*

VALUE OF A DETERMINANT OF ORDER 2 *We define*

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{21}a_{12}).$$

EXAMPLE 1 *Evaluate:*

$$(i) \begin{vmatrix} 6 & -3 \\ 7 & -2 \end{vmatrix} \qquad (ii) \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$$

SOLUTION (i) $\begin{vmatrix} 6 & -3 \\ 7 & -2 \end{vmatrix} = 6(-2) - 7(-3) = -12 + 21 = 9.$

$$\begin{aligned} (ii) \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} &= (x^2 - x + 1)(x + 1) - (x + 1)(x - 1) \\ &= (x^3 + 1) - (x^2 - 1) = (x^3 - x^2 + 2). \end{aligned}$$

EXAMPLE 2 If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x . [CBSE 2014]

SOLUTION We have $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

$$\Leftrightarrow (3x \times 4) - (-2) \times 7 = (8 \times 4) - (6 \times 7)$$

$$\Leftrightarrow 12x + 14 = 32 - 42$$

$$\Leftrightarrow 12x + 14 = -10$$

$$\Leftrightarrow 12x = -24 \Leftrightarrow x = -2.$$

Hence, $x = -2$.

EXAMPLE 3 If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, find the value of x . [CBSE 2013]

SOLUTION We have $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$

$$\Leftrightarrow (x+1)(x+2) - (x-3)(x-1) = (4 \times 3) - 1 \times (-1)$$

$$\Leftrightarrow (x^2 + 3x + 2) - (x^2 - 4x + 3) = 12 + 1$$

$$\Leftrightarrow 7x - 1 = 13 \Leftrightarrow 7x = 14 \Leftrightarrow x = 2.$$

Hence, $x = 2$.

EXAMPLE 4 Show that $\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix} = 1$.

SOLUTION We have $\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix}$

$$= (\sin 10^\circ)(\cos 80^\circ) - (\sin 80^\circ)(-\cos 10^\circ)$$

$$= (\sin 10^\circ \cos 80^\circ + \cos 10^\circ \sin 80^\circ)$$

$$= \sin (10^\circ + 80^\circ) \quad [\because \sin A \cos B + \cos A \sin B = \sin(A + B)]$$

$$= \sin 90^\circ = 1.$$

VALUE OF A DETERMINANT OF ORDER 3 OR MORE For finding the value of a determinant of order 3 or more, we need the following definitions.

MINOR OF a_{ij} IN $|A|$ The minor of an element a_{ij} in $|A|$ is defined as the value of the determinant obtained by deleting the i th row and j th column of $|A|$, and it is denoted by M_{ij} .

COFACTOR OF a_{ij} IN $|A|$ The cofactor C_{ij} of an element a_{ij} in $|A|$ is defined as $C_{ij} = (-1)^{i+j} \cdot M_{ij}$.

EXAMPLE 1 Find the minors and cofactors of the elements of the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

SOLUTION Let M_{ij} denote the minor of a_{ij} in Δ .

Now, a_{11} occurs in the 1st row and 1st column. So, in order to find the minor of a_{11} , we delete the 1st row and 1st column of Δ . The

minor M_{11} of a_{11} is given by $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}a_{33} - a_{32}a_{23})$.

Similarly, we have

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (a_{21}a_{33} - a_{31}a_{23});$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (a_{21}a_{32} - a_{31}a_{22});$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = (a_{12}a_{33} - a_{32}a_{13}).$$

Similarly, we may obtain the minor of each of the remaining elements.

Now, if we denote the cofactor of a_{ij} by C_{ij} then

$$C_{11} = (-1)^{1+1} \cdot M_{11} = M_{11} = (a_{22}a_{33} - a_{32}a_{23});$$

$$C_{12} = (-1)^{1+2} \cdot M_{12} = -M_{12} = (a_{31}a_{23} - a_{21}a_{33});$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = M_{13} = (a_{21}a_{32} - a_{31}a_{22});$$

$$C_{21} = (-1)^{2+1} \cdot M_{21} = -M_{21} = (a_{32}a_{13} - a_{12}a_{33}).$$

Similarly, the cofactor of each of the remaining elements of Δ can be determined.

EXAMPLE 2 Find the minor and cofactor of each element of $\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$.

SOLUTION The minors of the elements of Δ are given by

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = -12; \quad M_{12} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = 2; \quad M_{13} = \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix} = 23;$$

$$M_{21} = \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = -16; \quad M_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4; \quad M_{23} = \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} = 14;$$

$$M_{31} = \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} = -4; \quad M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = -6; \quad M_{33} = \begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} = 11.$$

So, the cofactors of the corresponding elements of Δ are

$$C_{11} = (-1)^{1+1} \cdot M_{11} = M_{11} = -12; \quad C_{12} = (-1)^{1+2} \cdot M_{12} = -M_{12} = -2;$$

$$\begin{aligned}
C_{13} &= (-1)^{1+3} \cdot M_{13} = M_{13} = 23; \quad C_{21} = (-1)^{2+1} \cdot M_{21} = -M_{21} = 16; \\
C_{22} &= (-1)^{2+2} \cdot M_{22} = M_{22} = -4; \quad C_{23} = (-1)^{2+3} \cdot M_{23} = -M_{23} = -14; \\
C_{31} &= (-1)^{3+1} \cdot M_{31} = M_{31} = -4; \quad C_{32} = (-1)^{3+2} \cdot M_{32} = -M_{32} = 6; \\
C_{33} &= (-1)^{3+3} \cdot M_{33} = M_{33} = 11.
\end{aligned}$$

Value of a Determinant

The value of a determinant is the sum of the products of elements of a row (or a column) with their corresponding cofactors.

We may expand a determinant by any arbitrarily chosen row or column.

Expansion of a Determinant

Expanding the given determinant by 1st row, we have

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11} \cdot (\text{its cofactor}) + a_{12} \cdot (\text{its cofactor}) + a_{13} \cdot (\text{its cofactor}) \\
&= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \quad [\because C_{12} = -M_{12}] \\
&= a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11} \cdot (a_{22}a_{33} - a_{32}a_{23}) - a_{12} \cdot (a_{21}a_{33} - a_{31}a_{23}) \\
&\quad + a_{13} \cdot (a_{21}a_{32} - a_{31}a_{22}).
\end{aligned}$$

We may expand it by any row or column.

REMARK 1 If we expand a determinant by any row or column using minors, we keep in view the following symbols for a determinant of order three:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

REMARK 2 If a row or a column of a determinant consists of all zeros, the value of the determinant is zero.

EXAMPLE 1 Evaluate $\Delta = \begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}$.

SOLUTION Expanding the given determinant by 1st row, we get

$$\begin{aligned}
\Delta &= 3 \cdot \begin{vmatrix} 2 & -3 \\ 1 & 7 \end{vmatrix} - 4 \cdot \begin{vmatrix} -6 & -3 \\ 8 & 7 \end{vmatrix} + 5 \cdot \begin{vmatrix} -6 & 2 \\ 8 & 1 \end{vmatrix} \\
&= 3 \cdot (14 + 3) - 4 \cdot (-42 + 24) + 5 \cdot (-6 - 16) = 13.
\end{aligned}$$

EXAMPLE 2 Expand the determinant $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

SOLUTION Expanding by 1st column, we get

$$\begin{aligned} \Delta &= a \cdot \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \cdot \begin{vmatrix} h & g \\ f & c \end{vmatrix} + g \cdot \begin{vmatrix} h & g \\ b & f \end{vmatrix} \\ &= a(bc - f^2) - h(ch - fg) + g(fh - bg) \\ &= abc - af^2 - ch^2 + fgh + fgh - bg^2 \\ &= (abc + 2fgh - af^2 - bg^2 - ch^2). \end{aligned}$$

Properties of Determinants

The properties of a determinant serve the purpose of a useful tool for finding its value. We will mention these properties and verify them for a third-order determinant.

THEOREM 1 The value of a determinant remains unchanged if its rows and columns are interchanged.

PROOF Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by

interchanging the rows and columns of Δ .

$$\begin{aligned} \text{Then, } \Delta' &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \cdot \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \cdot \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ &\quad \text{[expanded by 1st column]} \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - a_3b_2) \\ &= \Delta \quad \text{[expanded by 1st row].} \end{aligned}$$

COROLLARY If A is a square matrix then $|A'| = |A|$.

THEOREM 2 If two rows or columns of a determinant are interchanged then the determinant retains its absolute value but its sign is changed.

PROOF Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by

interchanging any two rows, say 1st and 3rd rows, of Δ . Then,

$$\begin{aligned}
\Delta' &= \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \\
&= a_3 \cdot \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_1 \cdot \begin{vmatrix} b_3 & c_3 \\ b_2 & c_2 \end{vmatrix} \\
&\quad \text{[expanded by 1st column]} \\
&= a_3(b_2c_1 - b_1c_2) - a_2(b_3c_1 - b_1c_3) + a_1(b_3c_2 - b_2c_3) \\
&= -a_1(b_2c_3 - b_3c_2) + a_2(b_1c_3 - b_3c_1) - a_3(b_1c_2 - b_2c_1) \\
&= -[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)] \\
&= -\Delta \quad \text{[expanded by 1st column].}
\end{aligned}$$

THEOREM 3 *If any two rows or columns of a determinant are identical then its value is zero.*

PROOF If we interchange the identical rows of the given determinant Δ then clearly there is no change in Δ . But, interchanging any two rows of a determinant changes its sign.

$$\therefore \Delta = -\Delta \Leftrightarrow 2\Delta = 0, \text{ i.e., } \Delta = 0.$$

THEOREM 4 *If each element of a row or a column of a determinant is multiplied by a constant k then the value of the new determinant is k times the value of the original determinant.*

PROOF Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by multiplying each element of a row, say the second row of Δ , by k . Then,

$$\begin{aligned}
\Delta' &= \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\
&= a_1 \cdot \begin{vmatrix} kb_2 & kc_2 \\ b_3 & c_3 \end{vmatrix} - ka_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ kb_2 & kc_2 \end{vmatrix} \\
&= a_1(kb_2c_3 - kb_3c_2) - ka_2(b_1c_3 - b_3c_1) + a_3(kb_1c_2 - kb_2c_1) \\
&= k[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)] = k\Delta.
\end{aligned}$$

COROLLARY *For a square matrix A of order n , $|kA| = k^n \cdot |A|$.*

THEOREM 5 *If any two rows or columns of a determinant are proportional then its value is zero.*

PROOF Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix}$

$$\begin{aligned}
&= k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} && \text{[by Theorem 4]} \\
&= k \times 0 = 0 && [\because \text{1st and 3rd rows are identical}].
\end{aligned}$$

THEOREM 6 *If each element of a row (or column) of a determinant is expressed as a sum of two or more terms then the determinant can be expressed as the sum of two or more determinants.*

PROOF Let $\Delta = \begin{vmatrix} a_1 + \alpha_1 & b_1 + \beta_1 & c_1 + \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Then, on expanding Δ by first row, we get

$$\begin{aligned}
\Delta &= (a_1 + \alpha_1)(b_2c_3 - b_3c_2) - (b_1 + \beta_1)(a_2c_3 - a_3c_2) \\
&\quad + (c_1 + \gamma_1)(a_2b_3 - a_3b_2) \\
&= [a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] \\
&\quad + [\alpha_1(b_2c_3 - b_3c_2) - \beta_1(a_2c_3 - a_3c_2) + \gamma_1(a_2b_3 - a_3b_2)] \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.
\end{aligned}$$

THEOREM 7 *If to any row or column of a determinant, a multiple of another row or column is added, the value of the determinant remains the same.*

PROOF Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Let $\Delta' = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Then, $\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad [\because \text{2nd det. is 0, its 1st and 3rd rows being proportional}]$$

$$= \Delta.$$

THEOREM 8 *The sum of the products of the elements of any row (or column) of a determinant with cofactors of the corresponding elements of any other row (or column) is zero.*

PROOF Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

Let us take the sum of the products of elements of first row with the cofactors of the corresponding elements of second row.

$$\begin{aligned} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} \\ &= a_{11} \cdot (-1)^{2+1} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \cdot (-1)^{2+2} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \\ & \quad + a_{13} \cdot (-1)^{2+3} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= -a_{11}(a_{12}a_{33} - a_{32}a_{13}) + a_{12}(a_{11}a_{33} - a_{31}a_{13}) - a_{13}(a_{11}a_{32} - a_{31}a_{12}) \\ &= 0. \end{aligned}$$

To Find the Value of a Determinant

The main theme behind the simplification of a determinant lies in obtaining the maximum possible number of zeros in a row (or a column) by using the above properties and then to expand the determinant by that row (or column). We denote the 1st, 2nd, 3rd rows of a determinant by R_1, R_2, R_3 respectively and the 1st, 2nd, 3rd columns by C_1, C_2, C_3 respectively. We shall also express the

- (i) interchange of the i th and j th rows by $R_i \leftrightarrow R_j$;
- (ii) addition of k times the elements of the j th row with the corresponding elements of the i th row by $R_i \rightarrow R_i + kR_j$.

We use similar notations for operations on columns, replacing R by C .

EXAMPLES

Operation	Notation
(i) Interchange of 2nd and 3rd rows.	(i) $R_2 \leftrightarrow R_3$
(ii) Interchange of 1st and 3rd columns	(ii) $C_1 \leftrightarrow C_3$
(iii) Multiplying each element of 2nd row by (-5)	(iii) $R_2 \rightarrow (-5)R_2$
(iv) Multiplying each element of 1st column by $\frac{1}{3}$.	(iv) $C_1 \rightarrow \frac{1}{3}C_1$
(v) Multiplying each element of 3rd row by 6 and adding it to the corresponding element of 2nd row.	(v) $R_2 \rightarrow R_2 + 6R_3$
(vi) Multiplying each element of 2nd column by (-3) and adding it to the corresponding element of 1st column.	(vi) $C_1 \rightarrow C_1 + (-3)C_2$

SOLVED EXAMPLES (Short-Answer Questions)

EXAMPLE 1 Evaluate $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$. [CBSE 2014C]

SOLUTION Given determinant

$$\begin{aligned}
 &= \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} \\
 &= \begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix} \quad [\text{applying } C_3 \rightarrow C_3 - 9C_2] \\
 &= 0 \quad [\because C_1 \text{ and } C_3 \text{ are identical}].
 \end{aligned}$$

EXAMPLE 2 Prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$. [CBSE 2009]

SOLUTION Given determinant

$$\begin{aligned}
 &= \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} \\
 &= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \quad [\text{applying } C_1 \rightarrow C_1 + C_2 + C_3] \\
 &= 0 \quad [\because C_1 \text{ consists of all zeros}].
 \end{aligned}$$

EXAMPLE 3 Prove that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$.

SOLUTION The given determinant

$$\begin{aligned}
 &= \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} \\
 &= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} \quad [\text{applying } C_3 \rightarrow C_3 + C_2]
 \end{aligned}$$

$$\begin{aligned}
 &= (a + b + c) \cdot \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} \quad [\text{taking } (a + b + c) \text{ common from } C_3] \\
 &= (a + b + c) \times 0 = 0 \quad [\because C_1 \text{ and } C_3 \text{ are identical}].
 \end{aligned}$$

EXAMPLE 4 Prove that $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0.$

SOLUTION The given determinant

$$\begin{aligned}
 &= \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} \\
 &= \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix} \quad [\text{applying } C_3 \rightarrow C_3 + C_2] \\
 &= (ab + bc + ca) \cdot \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} \quad [\text{taking } (ab + bc + ca) \text{ common from } C_3] \\
 &= (ab + bc + ca) \times 0 = 0 \quad [\because C_1 \text{ and } C_3 \text{ are identical}].
 \end{aligned}$$

EXAMPLE 5 Without expanding prove that $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0.$

SOLUTION The given determinant

$$\begin{aligned}
 &= \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} \quad [R_1 \rightarrow R_1 + R_2] \\
 &= (x + y + z) \cdot \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} \quad [\text{taking } (x + y + z) \text{ common from } R_1] \\
 &= (x + y + z) \times 0 = 0 \quad [\because R_1 \text{ and } R_3 \text{ are identical}].
 \end{aligned}$$

EXAMPLE 6 If ω is a complex cube root of unity, prove that

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0.$$

SOLUTION The given determinant

$$\begin{aligned} &= \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \\ &= \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \quad [R_1 \rightarrow R_1 + R_2 + R_3] \\ &= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \quad [\because 1+\omega+\omega^2=0] \\ &= 0 \quad [\because R_1 \text{ consists of all zeros}]. \end{aligned}$$

EXAMPLE 7 Show that $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix} = 0.$ **[CBSE 2009]**

SOLUTION The given determinant

$$\begin{aligned} &= \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix} \\ &= (3x) \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix} \quad [\text{taking } 3x \text{ common from } R_3] \\ &= (3x) \times 0 = 0 \quad [\because R_1 \text{ and } R_3 \text{ are identical}]. \end{aligned}$$

EXAMPLE 8 Prove that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy.$

SOLUTION The given determinant

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$$

$$\begin{aligned}
 &= \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix} \quad [\text{applying } C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1] \\
 &= 1 \cdot \begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} = xy \quad [\text{expanding by } R_1].
 \end{aligned}$$

EXAMPLE 9 Prove that $\begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$ is independent of θ .

SOLUTION Let the given determinant be Δ . Then, expanding by first row, we get:

$$\begin{aligned}
 \Delta &= \begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix} \\
 &= x(-x^2 - 1) - \sin \theta(-x \sin \theta - \cos \theta) + \cos \theta(-\sin \theta + x \cos \theta) \\
 &= -x^3 - x + x(\sin^2 \theta + \cos^2 \theta) = -x^3 - x + x = -x^3, \\
 &\quad \text{which is independent of } \theta.
 \end{aligned}$$

EXAMPLE 10 Prove that $\begin{vmatrix} 0 & \sin \alpha & -\cos \alpha \\ -\sin \alpha & 0 & \sin \beta \\ \cos \alpha & -\sin \beta & 0 \end{vmatrix} = 0$.

SOLUTION Let the given determinant be Δ . Then, expanding by first row, we get:

$$\begin{aligned}
 \Delta &= (-\sin \alpha) \cdot \begin{vmatrix} -\sin \alpha & \sin \beta \\ \cos \alpha & 0 \end{vmatrix} + (-\cos \alpha) \cdot \begin{vmatrix} -\sin \alpha & 0 \\ \cos \alpha & -\sin \beta \end{vmatrix} \\
 &= (-\sin \alpha) \cdot \{0 - \cos \alpha \sin \beta\} + (-\cos \alpha) \cdot (\sin \alpha \sin \beta - 0) \\
 &= (-\sin \alpha)(-\cos \alpha \sin \beta) + (-\cos \alpha)(\sin \alpha \sin \beta) \\
 &= (\sin \alpha \cos \alpha \sin \beta - \sin \alpha \cos \alpha \sin \beta) = 0.
 \end{aligned}$$

Hence, $\Delta = 0$.

EXAMPLE 11 Prove that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$.

SOLUTION Let the given determinant be Δ . Then, applying $R_2 \rightarrow R_2 - R_1$, we get:

$$\Delta = \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix} = 0 \quad [\because R_2 \text{ and } R_3 \text{ are identical}].$$

EXAMPLE 12 Without expanding the determinant, prove that

$$\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix} = 0.$$

SOLUTION Let the given determinant be Δ . Then, applying $C_3 \rightarrow C_3 - C_2$, we get:

$$\begin{aligned} \Delta &= \begin{vmatrix} a+b & 2a+b & a \\ 2a+b & 3a+b & a \\ 4a+b & 5a+b & a \end{vmatrix} \\ &= \begin{vmatrix} a+b & a & a \\ 2a+b & a & a \\ 4a+b & a & a \end{vmatrix} \quad [\text{applying } C_2 \rightarrow C_2 - C_1] \\ &= 0 \quad [\because C_2 \text{ and } C_3 \text{ are identical}]. \end{aligned}$$

Hence, $\Delta = 0$.

ILLUSTRATIVE EXAMPLES [Long-Answer Questions]

EXAMPLE 1 Evaluate $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$.

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix} \\ &= \begin{vmatrix} 46 & 21 & 219 \\ 42 & 27 & 198 \\ 38 & 17 & 181 \end{vmatrix} \quad [C_1 \rightarrow (C_1 - C_3) \text{ and } C_2 \rightarrow (C_2 - C_3)] \\ &= \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix} \quad [C_1 \rightarrow (C_1 - 2C_2) \text{ and } C_3 \rightarrow (C_3 - 10C_2)] \\ &= \begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix} \quad [R_1 \rightarrow (R_1 - R_3) \text{ and } R_2 \rightarrow (R_2 + 3R_3)] \\ &= 2(39) \begin{vmatrix} 0 & 2 & -1 \\ 0 & 2 & -1 \\ 4 & 17 & 11 \end{vmatrix} \\ &\quad [\text{taking 2 common from } R_1 \text{ and 39 common from } R_2] \\ &= (78 \times 0) = 0 \quad [\because R_1 \text{ and } R_2 \text{ are identical}]. \end{aligned}$$

EXAMPLE 2 Without expanding, prove that

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos (\alpha + \delta) \\ \sin \beta & \cos \beta & \cos (\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos (\gamma + \delta) \end{vmatrix} = 0. \quad [\text{CBSE 2007}]$$

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \cos \delta - \sin \alpha \sin \delta \\ \sin \beta & \cos \beta & \cos \beta \cos \delta - \sin \beta \sin \delta \\ \sin \gamma & \cos \gamma & \cos \gamma \cos \delta - \sin \gamma \sin \delta \end{vmatrix} \\ &= \begin{vmatrix} \sin \alpha & \cos \alpha & 0 \\ \sin \beta & \cos \beta & 0 \\ \sin \gamma & \cos \gamma & 0 \end{vmatrix} \quad [C_3 \rightarrow C_3 + (\sin \delta)C_1 - (\cos \delta)C_2] \\ &= 0 \quad [\text{expanded by } C_3]. \end{aligned}$$

Hence, $\Delta = 0$.

EXAMPLE 3 Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$. [CBSE 2012]

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} \\ &= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix} \quad [R_2 \rightarrow R_2 - 2R_1 \text{ and } R_3 \rightarrow R_3 - 3R_1] \\ &= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix} \quad [R_3 \rightarrow R_3 - 3R_2] \\ &= a \cdot \begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} \quad [\text{expanding along } C_1] \\ &= a \cdot (a^2 - 0) = a^3. \end{aligned}$$

Hence, $\Delta = a^3$.

EXAMPLE 4 Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$. [CBSE 2011, '11C]

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} \\
 &= (abc) \cdot \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix} \left[\begin{array}{l} \text{taking out } a, b, c \text{ common from} \\ C_1, C_2, C_3 \text{ respectively} \end{array} \right] \\
 &= \begin{vmatrix} -a & 0 & 0 \\ b & 0 & 2b \\ c & 2c & 0 \end{vmatrix} \quad [C_2 \rightarrow (C_2 + C_1) \text{ and } C_3 \rightarrow (C_3 + C_1)] \\
 &= (abc) \cdot (-a) \begin{vmatrix} 0 & 2b \\ 2c & 0 \end{vmatrix} = (abc)(-a)(-4bc) \\
 &= 4a^2b^2c^2.
 \end{aligned}$$

EXAMPLE 5 Using properties of determinants, prove that

$$\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & cb^2 & 0 \end{vmatrix} = 2a^3b^3c^3.$$

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & cb^2 & 0 \end{vmatrix} \\
 &= (a^2b^2c^2) \cdot \begin{vmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{vmatrix} \left[\begin{array}{l} \text{taking } a^2, b^2, c^2 \text{ common from} \\ C_1, C_2 \text{ and } C_3 \text{ respectively} \end{array} \right] \\
 &= (a^3b^3c^3) \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \left[\begin{array}{l} \text{taking } a, b, c \text{ common from} \\ R_1, R_2 \text{ and } R_3 \text{ respectively} \end{array} \right] \\
 &= (a^3b^3c^3) \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} \quad [R_3 \rightarrow R_3 - R_2] \\
 &= (a^3b^3c^3) \cdot (-1) \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \quad [\text{expanded by } C_1] \\
 &= -(a^3b^3c^3)(-1-1) = 2a^3b^3c^3.
 \end{aligned}$$

Hence, $\Delta = 2a^3b^3c^3$.

EXAMPLE 6 Using properties of determinants, prove that

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3. \quad [\text{CBSE 2009, '14}]$$

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} \\ &= \begin{vmatrix} x+y & x & x \\ 3x+2y & 2x & 0 \\ 7x+5y & 5x & 0 \end{vmatrix} \quad [R_2 \rightarrow (R_2 - 2R_1) \text{ and } R_3 \rightarrow (R_3 - 3R_1)] \\ &= x \cdot \begin{vmatrix} 3x+2y & 2x \\ 7x+5y & 5x \end{vmatrix} \quad [\text{expanded by } C_3] \\ &= x^2 \cdot \begin{vmatrix} 3x+2y & 2 \\ 7x+5y & 5 \end{vmatrix} \quad [\text{taking } x \text{ common from } C_2] \\ &= x^2 \cdot [5(3x+2y) - 2(7x+5y)] = (x^2 \cdot x) = x^3. \end{aligned}$$

Hence, $\Delta = x^3$.

EXAMPLE 7 Prove that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad [\text{applying } R_2 \rightarrow (R_2 - R_1) \text{ and } R_3 \rightarrow (R_3 - R_1)] \\ &= (b-a)(c-a) \cdot \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} \\ &\quad [\text{taking } (b-a) \text{ common from } R_2 \text{ and } (c-a) \text{ common from } R_3] \\ &= (b-a)(c-a) \times 1 \cdot \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} \quad [\text{expanded by } C_1] \\ &= (b-a)(c-a)\{(c+a) - (b+a)\} \\ &= (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a). \end{aligned}$$

Hence, $\Delta = (a-b)(b-c)(c-a)$.

EXAMPLE 8 Prove that
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

[CBSE 2009C, '11, '12, '13C]

SOLUTION Let the value of the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix} \quad [\text{applying } C_1 \rightarrow (C_1 - C_3) \text{ and } C_2 \rightarrow (C_2 - C_3)] \\ &= (a-c)(b-c) \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ac+c^2 & b^2+bc+c^2 & c^3 \end{vmatrix} \\ &\quad [\text{taking out } (a-c) \text{ and } (b-c) \text{ common from } C_1 \text{ and } C_2] \\ &= (a-c)(b-c) \cdot 1 \cdot \begin{vmatrix} 1 & 1 \\ a^2+ac+c^2 & b^2+bc+c^2 \end{vmatrix} \quad [\text{expanded by } R_1] \\ &= (a-c)(b-c) \cdot [(b^2+bc+c^2) - (a^2+ac+c^2)] \\ &= (a-c)(b-c)[(b^2-a^2) + (bc-ac)] \\ &= (a-c)(b-c)[(b^2-a^2) + (b-a)c] \\ &= (a-c)(b-c)(b-a)(b+a+c) \\ &= (a-b)(b-c)(c-a)(a+b+c). \end{aligned}$$

Hence, $\Delta = (a-b)(b-c)(c-a)(a+b+c)$.

EXAMPLE 9 Prove that
$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma).$$

[CBSE 2007C, '08, '10C, '11C, '12C]

SOLUTION Let the value of the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} \\ &= \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix} \quad [\text{applying } R_3 \rightarrow (R_3 + R_1)] \end{aligned}$$

$$\begin{aligned}
&= (\alpha + \beta + \gamma) \cdot \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix} \quad [\text{taking } (\alpha + \beta + \gamma) \text{ common from } R_3] \\
&= (\alpha + \beta + \gamma) \cdot \begin{vmatrix} \alpha - \gamma & \beta - \gamma & \gamma \\ \alpha^2 - \gamma^2 & \beta^2 - \gamma^2 & \gamma^2 \\ 0 & 0 & 1 \end{vmatrix} \\
&\quad [\text{applying } C_1 \rightarrow (C_1 - C_3) \text{ and } C_2 \rightarrow (C_2 - C_3)] \\
&= (\alpha + \beta + \gamma)(\alpha - \gamma)(\beta - \gamma) \cdot \begin{vmatrix} 1 & 1 & \gamma \\ \alpha + \gamma & \beta + \gamma & \gamma^2 \\ 0 & 0 & 1 \end{vmatrix} \\
&\quad [\text{taking } (\alpha - \gamma) \text{ common from } C_1 \text{ and } (\beta - \gamma) \text{ common from } C_2] \\
&= (\alpha + \beta + \gamma)(\alpha - \gamma)(\beta - \gamma) \cdot 1 \cdot \begin{vmatrix} 1 & 1 \\ \alpha + \gamma & \beta + \gamma \end{vmatrix} \quad [\text{expanded by } R_3] \\
&= (\alpha + \beta + \gamma)(\alpha - \gamma)(\beta - \gamma)[(\beta + \gamma) - (\alpha + \gamma)] \\
&= (\alpha + \beta + \gamma)(\alpha - \gamma)(\beta - \gamma)(\beta - \alpha) \\
&= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma).
\end{aligned}$$

Hence, $\Delta = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma)$.

EXAMPLE 10 Using properties of determinants, show that

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z). \quad [\text{CBSE 2003, '14}]$$

SOLUTION Let the value of the given determinant be Δ . Then,

$$\begin{aligned}
\Delta &= \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} \\
&= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix} \quad [C_1 \rightarrow C_1 + C_2 + C_3] \\
&= (a+x+y+z) \cdot \begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix} \\
&\quad [\text{taking } (a+x+y+z) \text{ common from } C_1] \\
&= (a+x+y+z) \cdot \begin{vmatrix} 1 & y & z \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} \quad [R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1] \\
&= (a+x+y+z) \cdot 1 \cdot \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \quad [\text{expanded by } C_1] \\
&= a^2(a+x+y+z).
\end{aligned}$$

Hence, $\Delta = a^2(a+x+y+z)$.

EXAMPLE 11 Using properties of determinants, prove that

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1 - x^3)^2. \quad [\text{CBSE 2008C, '13}]$$

SOLUTION Let the value of the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix} \quad [\text{applying } C_1 \rightarrow (C_1 + C_2 + C_3)] \\ &= (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix} \quad [\text{taking } (1+x+x^2) \text{ common from } C_1] \\ &= (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x(1-x) \\ 0 & x(x-1) & (1-x)(1+x) \end{vmatrix} \\ &\quad [\text{applying } R_2 \rightarrow (R_2 - R_1) \text{ and } R_3 \rightarrow (R_3 - R_1)] \\ &= (1+x+x^2)(1-x)^2 \cdot \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & -x & (1+x) \end{vmatrix} \\ &\quad [\text{taking } (1-x) \text{ common from each of } R_2 \text{ and } R_3] \\ &= (1+x+x^2)(1-x)^2 \cdot 1 \cdot \begin{vmatrix} 1 & x \\ -x & (1+x) \end{vmatrix} \quad [\text{expanded by } C_1] \\ &= (1+x+x^2)(1-x)^2 \cdot (1+x+x^2) \\ &= \{(1-x)(1+x+x^2)\}^2 = (1-x^3)^2. \end{aligned}$$

Hence, $\Delta = (1 - x^3)^2$.

EXAMPLE 12 Using the properties of determinants, prove that

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3. \quad [\text{CBSE 2014}]$$

SOLUTION Let the value of the given determinant be Δ . Then,

$$\Delta = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

$$\begin{aligned}
&= \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x+y+z & x+y+z & x+y+z \end{vmatrix} \quad [\text{applying } R_3 \rightarrow (R_1 + R_2 + R_3)] \\
&= (x+y+z) \cdot \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ 1 & 1 & 1 \end{vmatrix} \\
&\quad [\text{taking } (x+y+z) \text{ common from } R_3] \\
&= (x+y+z) \cdot \begin{vmatrix} 0 & -(x+y+z) & 2y \\ (x+y+z) & (x+y+z) & z-x-y \\ 0 & 0 & 1 \end{vmatrix} \\
&\quad [\text{applying } C_1 \rightarrow (C_1 - C_3) \text{ and } C_2 \rightarrow (C_2 - C_3)] \\
&= (x+y+z) \cdot 1 \cdot \begin{vmatrix} 0 & -(x+y+z) \\ (x+y+z) & (x+y+z) \end{vmatrix} \quad [\text{expanded by } R_3] \\
&= (x+y+z)\{0 + (x+y+z)^2\} = (x+y+z)(x+y+z)^2 = (x+y+z)^3. \\
&\text{Hence, } \Delta = (x+y+z)^3.
\end{aligned}$$

EXAMPLE 13 Using properties of determinants, prove that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca). \quad [\text{CBSE 2006, '13}]$$

SOLUTION Let the value of the given determinant be Δ . Then,

$$\begin{aligned}
\Delta &= \begin{vmatrix} 3a & -a+b & -a+c \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{vmatrix} \\
&= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & c-b \\ a+b+c & b-c & 3c \end{vmatrix} \quad [C_1 \rightarrow (C_1 + C_2 + C_3)] \\
&= (a+b+c) \cdot \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & c-b \\ 1 & b-c & 3c \end{vmatrix} \\
&\quad [\text{taking } (a+b+c) \text{ common from } C_1] \\
&= (a+b+c) \cdot \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix} \\
&\quad [R_2 \rightarrow (R_2 - R_1) \text{ and } R_3 \rightarrow (R_3 - R_1)] \\
&= (a+b+c) \cdot 1 \cdot \begin{vmatrix} 2b+a & a-b \\ a-c & 2c+a \end{vmatrix} \quad [\text{expanded by } C_1] \\
&= (a+b+c)[(2b+a)(2c+a) - (a-c)(a-b)]
\end{aligned}$$

$$\begin{aligned}
 &= (a + b + c)[(4bc + 2ab + 2ac + a^2) - (a^2 - ab - ac + bc)] \\
 &= 3(a + b + c)(ab + bc + ca).
 \end{aligned}$$

Hence, $\Delta = 3(a + b + c)(ab + bc + ca)$.

EXAMPLE 14 Prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(a+b+c)(ab+bc+ca-a^2-b^2-c^2).$$

[CBSE 2004, '09C]

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \\
 &= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} [R_1 \rightarrow (R_1 + R_2 + R_3)] \\
 &= 2(a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \\
 &\quad \text{[taking } (a+b+c) \text{ common from } R_1] \\
 &= 2(a+b+c) \cdot \begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix} \begin{bmatrix} C_2 \rightarrow (C_2 - C_1) \text{ and} \\ C_3 \rightarrow (C_3 - C_1) \end{bmatrix} \\
 &= 2(a+b+c) \cdot 1 \cdot \begin{vmatrix} b-c & b-a \\ c-a & c-b \end{vmatrix} \text{ [expanded by } R_1] \\
 &= 2(a+b+c) \cdot [(b-c)(c-b) - (c-a)(b-a)] \\
 &= 2(a+b+c)(ab+bc+ca-a^2-b^2-c^2).
 \end{aligned}$$

Hence, $\Delta = 2(a + b + c)(ab + bc + ca - a^2 - b^2 - c^2)$.

EXAMPLE 15 Using properties of determinants, prove that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3. \text{ [CBSE 2001, '04, '06C, '07]}$$

SOLUTION Let the value of the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \\
 &= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} [R_1 \rightarrow (R_1 + R_2 + R_3)]
 \end{aligned}$$

$$\begin{aligned}
&= (a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \\
&\quad \text{[taking } (a+b+c) \text{ common from } R_1] \\
&= (a+b+c) \cdot \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix} \\
&\quad [C_2 \rightarrow (C_2 - C_1) \text{ and } C_3 \rightarrow (C_3 - C_1)] \\
&= (a+b+c) \cdot 1 \cdot \begin{vmatrix} -(a+b+c) & 0 \\ 0 & -(a+b+c) \end{vmatrix} \quad \text{[expanded by } R_1] \\
&= (a+b+c)(a+b+c)^2 = (a+b+c)^3.
\end{aligned}$$

Hence, $\Delta = (a+b+c)^3$.

EXAMPLE 16 Prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$$

[CBSE 2008, '12C, '14C]

SOLUTION Let the value of the given determinant be Δ . Then,

$$\begin{aligned}
\Delta &= \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} \\
&= \begin{vmatrix} a+b+c & -(a+b+c) & 0 \\ c & b+c+2a & b \\ 0 & -(a+b+c) & (a+b+c) \end{vmatrix} \quad \left\{ \begin{array}{l} \text{by } R_1 \rightarrow (R_1 - R_2) \\ \text{and } R_3 \rightarrow (R_3 - R_2) \end{array} \right\} \\
&= (a+b+c)^2 \cdot \begin{vmatrix} 1 & -1 & 0 \\ c & b+c+2a & b \\ 0 & -1 & 1 \end{vmatrix} \\
&\quad \text{[taking } (a+b+c) \text{ common from each one of } R_1 \text{ and } R_3] \\
&= (a+b+c)^2 \cdot \begin{vmatrix} 1 & -1 & 0 \\ 0 & b+2c+2a & b \\ 0 & -1 & 1 \end{vmatrix} \quad [R_2 \rightarrow R_2 - cR_1] \\
&= (a+b+c)^2 \cdot 1 \cdot \begin{vmatrix} b+2c+2a & b \\ -1 & 1 \end{vmatrix} \quad \text{[expanded by } C_1] \\
&= (a+b+c)^2 \cdot (b+2c+2a+b) = 2(a+b+c)^3.
\end{aligned}$$

Hence, $\Delta = 2(a+b+c)^3$.

EXAMPLE 17 Using properties of determinants, prove that

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca). \quad [\text{CBSE 2011C, '13C}]$$

SOLUTION Let the value of the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} \\ &= \begin{vmatrix} a-c & b-c & c \\ a^2-c^2 & b^2-c^2 & c^2 \\ b(c-a) & a(c-b) & ab \end{vmatrix} \quad [C_1 \rightarrow (C_1 - C_3), C_2 \rightarrow (C_2 - C_3)] \\ &= (a-c)(b-c) \cdot \begin{vmatrix} 1 & 1 & c \\ a+c & b+c & c^2 \\ -b & -a & ab \end{vmatrix} \\ &\quad [\text{taking } (a-c) \text{ common from } C_1 \text{ and } (b-c) \text{ common from } C_2] \\ &= (a-c)(b-c) \cdot \begin{vmatrix} 1 & 0 & 0 \\ a+c & b-a & -ca \\ -b & b-a & (a+c)b \end{vmatrix} \\ &\quad [C_2 \rightarrow (C_2 - C_1), C_3 \rightarrow (C_3 - cC_1)] \\ &= (a-c)(b-c) \cdot 1 \cdot \begin{vmatrix} b-a & -ca \\ b-a & (a+c)b \end{vmatrix} \quad \{\text{expanded by } R_1\} \\ &= (a-c)(b-c) \cdot (b-a) \begin{vmatrix} 1 & -ca \\ 1 & (a+c)b \end{vmatrix} \\ &= (a-b)(b-c)(c-a)(ab+bc+ca). \end{aligned}$$

Hence, $\Delta = (a-b)(b-c)(c-a)(ab+bc+ca)$.

EXAMPLE 18 Using properties of determinants, prove that

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc). \quad [\text{CBSE 2006, '09, '12C}]$$

SOLUTION Let the value of the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} \\ &= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix} \quad [C_1 \rightarrow (C_1 + C_2 + C_3)] \end{aligned}$$

$$\begin{aligned}
&= (a+b+c) \cdot \begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix} \\
&\quad \text{[taking } (a+b+c) \text{ common from } C_1] \\
&= (a+b+c) \cdot \begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix} \quad [R_3 \rightarrow R_3 - 2R_1] \\
&= (a+b+c) \cdot 1 \cdot \begin{vmatrix} b-c & c-a \\ c+a-2b & a+b-2c \end{vmatrix} \quad \text{[expanded by } C_1] \\
&= (a+b+c) \cdot \begin{vmatrix} b-c & c-a \\ a-b & b-c \end{vmatrix} \quad [R_2 \rightarrow R_2 + R_1] \\
&= (a+b+c) \cdot [(b-c)^2 - (a-b)(c-a)] \\
&= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \\
&= (a^3 + b^3 + c^3 - 3abc).
\end{aligned}$$

Hence, $\Delta = (a^3 + b^3 + c^3 - 3abc)$.

EXAMPLE 19 Prove that $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$.

SOLUTION Given determinant $= \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$

$$\begin{aligned}
&= \begin{vmatrix} 0 & -2x & -2x \\ z & z+x & x \\ y & x & x+y \end{vmatrix} \quad [R_1 \rightarrow R_1 - (R_2 + R_3)] \\
&= (-2x) \cdot \begin{vmatrix} 0 & 1 & 1 \\ z & z+x & x \\ y & x & x+y \end{vmatrix} \\
&\quad \text{[taking } (-2x) \text{ common from } R_1] \\
&= (-2x) \cdot \begin{vmatrix} 0 & 0 & 1 \\ z & z & x \\ y & -y & x+y \end{vmatrix} \quad [C_2 \rightarrow (C_2 - C_3)] \\
&= (-2x) \cdot 1 \cdot \begin{vmatrix} z & z \\ y & -y \end{vmatrix} \quad \text{[expanded by } R_1] \\
&= (-2x) \cdot 1 \cdot (-yz - yz) = (-2x)(-2yz) = 4xyz.
\end{aligned}$$

EXAMPLE 20 Prove that
$$\begin{vmatrix} a^2 & bc & c^2 + ac \\ a^2 + ab & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix} = 4a^2b^2c^2. \quad [\text{CBSE 2014}]$$

SOLUTION

$$\begin{vmatrix} a^2 & bc & c^2 + ac \\ a^2 + ab & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix}$$

$$= (abc) \cdot \begin{vmatrix} a & c & c+a \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

[taking a, b, c common from C_1, C_2 and C_3 respectively]

$$= (abc) \cdot \begin{vmatrix} a & c & c+a \\ 0 & -2c & -2c \\ b & b+c & c \end{vmatrix} \quad [R_2 \rightarrow R_2 - (R_1 + R_3)]$$

$$= (abc) \cdot \begin{vmatrix} a & -a & c+a \\ 0 & 0 & -2c \\ b & b & c \end{vmatrix} \quad [C_2 \rightarrow C_2 - C_3]$$

$$= (abc)(2c) \cdot \begin{vmatrix} a & -a \\ b & b \end{vmatrix} \quad [\text{expanded by } R_2]$$

$$= 2abc^2 \cdot (ab + ab) = 2abc^2(2ab)$$

$$= 4a^2b^2c^2.$$

EXAMPLE 21 If a, b, c are positive and unequal, show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

SOLUTION The given determinant

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} \quad [C_1 \rightarrow (C_1 + C_2 + C_3)]$$

$$= (a+b+c) \cdot \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$$\begin{aligned}
&= (a+b+c) \cdot \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} [R_2 \rightarrow (R_2 - R_1) \text{ and } R_3 \rightarrow (R_3 - R_1)] \\
&= (a+b+c) \cdot [(c-b)(b-c) - (a-b)(a-c)] \quad [\text{expanded by } C_1] \\
&= (a+b+c) \cdot (-a^2 - b^2 - c^2 + ab + bc + ca) \\
&= -\frac{1}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) \\
&= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2], \text{ which is negative} \\
&\quad [\because (a+b+c) > 0, (a-b)^2 > 0, (b-c)^2 > 0 \text{ and } (c-a)^2 > 0].
\end{aligned}$$

EXAMPLE 22 Evaluate $\begin{vmatrix} {}^mC_1 & {}^mC_2 & {}^mC_3 \\ {}^nC_1 & {}^nC_2 & {}^nC_3 \\ {}^pC_1 & {}^pC_2 & {}^pC_3 \end{vmatrix}$.

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
\Delta &= \begin{vmatrix} m & \frac{1}{2}m(m-1) & \frac{1}{6}m(m-1)(m-2) \\ n & \frac{1}{2}n(n-1) & \frac{1}{6}n(n-1)(n-2) \\ p & \frac{1}{2}p(p-1) & \frac{1}{6}p(p-1)(p-2) \end{vmatrix} \\
&= \left(\frac{1}{2} \times \frac{1}{6} \times mnp \right) \cdot \begin{vmatrix} 1 & (m-1) & (m-1)(m-2) \\ 1 & (n-1) & (n-1)(n-2) \\ 1 & (p-1) & (p-1)(p-2) \end{vmatrix} \\
&= \frac{1}{12} \cdot mnp \cdot \begin{vmatrix} 1 & m-1 & (m-1)(m-2) \\ 0 & n-m & (n-m)(n+m-3) \\ 0 & p-m & (p-m)(p+m-3) \end{vmatrix} \\
&\quad [R_2 \rightarrow (R_2 - R_1) \text{ and } R_3 \rightarrow (R_3 - R_1)] \\
&= \frac{1}{12} \cdot (mnp)(n-m)(p-m) \cdot \begin{vmatrix} 1 & m-1 & (m-1)(m-2) \\ 0 & 1 & (n+m-3) \\ 0 & 1 & (p+m-3) \end{vmatrix} \\
&\quad [\text{taking } (n-m) \text{ common from } R_2 \text{ and } (p-m) \text{ common from } R_3] \\
&= \frac{1}{12} \cdot (mnp)(n-m)(p-m) \cdot 1 \cdot \begin{vmatrix} 1 & (n+m-3) \\ 1 & (p+m-3) \end{vmatrix} \\
&= \frac{1}{12} \cdot (mnp)(n-m)(p-m)[(p+m-3) - (n+m-3)] \\
&= \frac{1}{12} \cdot (mnp)(n-m)(p-m)(p-n)
\end{aligned}$$

$$= \frac{1}{12} \cdot (mnp)(m-n)(n-p)(p-m).$$

$$\text{Hence, } \Delta = \frac{1}{12} \cdot (mnp)(m-n)(n-p)(p-m).$$

EXAMPLE 23 Prove that
$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

[CBSE 2010, '10C]

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} \\ &= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix} \\ &\quad [C_1 \rightarrow C_1 - C_3 \text{ and } C_2 \rightarrow C_2 - C_3] \\ &= \begin{vmatrix} (a+b+c)(b+c-a) & 0 & a^2 \\ 0 & (a+b+c)(c+a-b) & b^2 \\ (a+b+c)(c-a-b) & (a+b+c)(c-a-b) & (a+b)^2 \end{vmatrix} \\ &= (a+b+c)^2 \cdot \begin{vmatrix} (b+c-a) & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} \\ &\quad [\text{taking } (a+b+c) \text{ common from } C_1 \text{ and } C_2 \text{ both}] \\ &= (a+b+c)^2 \cdot \begin{vmatrix} (b+c-a) & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad [R_3 \rightarrow R_3 - (R_1 + R_2)] \\ &= (a+b+c)^2 [(b+c-a)\{(c+a-b) \cdot 2ab + 2ab^2\} + a^2\{0 + 2b(c+a-b)\}] \\ &= (a+b+c)^2 [(b+c-a) \cdot 2ab\{(c+a-b+b)\} + 2a^2b(c+a-b)] \\ &= 2ab(a+b+c)^2 \{(b+c-a)(c+a) + a(c+a-b)\} \\ &= 2ab(a+b+c)^2 \cdot \{bc + ab + c^2 + ac - ac - a^2 + ac + a^2 - ab\} \\ &= 2ab(a+b+c)^2 \{bc + c^2 + ac\} = 2abc(a+b+c)^3. \end{aligned}$$

$$\text{Hence, } \Delta = 2abc(a+b+c)^3.$$

TYPE: EXPRESSING A GIVEN DET. AS SUM OF TWO DETERMINANTS

EXAMPLE 24 Prove that
$$\begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix} = (a+b+c)(a-c)^2.$$
 [CBSE 2005C, '07]

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix} \\
 &= \begin{vmatrix} b & a & b \\ c & c & a \\ a & b & c \end{vmatrix} + \begin{vmatrix} c & a & b \\ a & c & a \\ b & b & c \end{vmatrix} \\
 &= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c & c & a \\ a & b & c \end{vmatrix} + \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & c & a \\ b & b & c \end{vmatrix} \\
 &\quad [R_1 \rightarrow (R_1 + R_2 + R_3) \text{ in each determinant}] \\
 &= (a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ c & c & a \\ a & b & c \end{vmatrix} + (a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & c & a \\ b & b & c \end{vmatrix} \\
 &\quad [\text{taking out } (a+b+c) \text{ common from } R_1 \text{ in each determinant}] \\
 &= (a+b+c) \cdot \begin{vmatrix} 1 & 0 & 0 \\ c & 0 & a-c \\ a & b-a & c-a \end{vmatrix} + (a+b+c) \cdot \begin{vmatrix} 1 & 0 & 0 \\ a & c-a & 0 \\ b & 0 & c-b \end{vmatrix} \\
 &\quad [C_2 \rightarrow (C_2 - C_1) \text{ and } C_3 \rightarrow (C_3 - C_1) \text{ in each}] \\
 &= (a+b+c) \cdot 1 \cdot \begin{vmatrix} 0 & a-c \\ b-a & c-a \end{vmatrix} + (a+b+c) \cdot 1 \cdot \begin{vmatrix} c-a & 0 \\ 0 & c-b \end{vmatrix} \\
 &\quad [\text{each det. expanded by } R_1] \\
 &= (a+b+c) \cdot [0 - (b-a)(a-c)] + (a+b+c)(c-a)(c-b) \\
 &= (a+b+c)(a-b)(a-c) - (a+b+c)(a-c)(c-b) \\
 &= (a+b+c)(a-c)\{(a-b) - (c-b)\} \\
 &= (a+b+c)(a-c)^2.
 \end{aligned}$$

Hence, $\Delta = (a+b+c)(a-c)^2$.

EXAMPLE 25 Prove that $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0$. [CBSE 2005C]

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} \\
 &= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 1 & b & -ca \\ 1 & c & -ab \end{vmatrix}
 \end{aligned}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 0 & b-a & c(b-a) \\ 0 & c-a & b(c-a) \end{vmatrix}$$

[$R_2 \rightarrow (R_2 - R_1)$ and $R_3 \rightarrow (R_3 - R_1)$ in each determinant]

$$\begin{aligned} &= (b-a)(c-a) \cdot \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} + (b-a)(c-a) \cdot \begin{vmatrix} 1 & a & -bc \\ 0 & 1 & c \\ 0 & 1 & b \end{vmatrix} \\ &= (b-a)(c-a) \cdot 1 \cdot \{(c+a) - (b+a)\} + (b-a)(c-a) \cdot 1 \cdot (b-c) \\ &= (b-a)(c-a)(c-b) + (b-a)(c-a)(b-c) \\ &= (a-b)(b-c)(c-a) - (a-b)(b-c)(c-a) = 0. \end{aligned}$$

Hence, $\Delta = 0$.

EXAMPLE 26 If $x \neq y \neq z$ and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ then prove that $xyz = -1$.

[CBSE 2008, '09C, '11]

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned} \Delta &= \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} \\ &= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (xyz) \cdot \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \\ &= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (xyz)(-1)^2 \cdot \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} \end{aligned}$$

[interchanging the columns of the 2nd det. twice]

$$\begin{aligned} &= (1+xyz) \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} \\ &= (1+xyz) \begin{vmatrix} x & x^2 & 1 \\ (y-x) & (y^2-x^2) & 0 \\ (z-x) & (z^2-x^2) & 0 \end{vmatrix} [R_2 \rightarrow (R_2 - R_1), R_3 \rightarrow (R_3 - R_1)] \\ &= (1+xyz)(y-x)(z-x) \begin{vmatrix} x & x^2 & 1 \\ 1 & y+z & 0 \\ 1 & z+x & 0 \end{vmatrix} \end{aligned}$$

$$\begin{aligned}
 &= (1 + xyz)(y - x)(z - x) \cdot 1 \cdot \begin{vmatrix} 1 & y + x \\ 1 & z + x \end{vmatrix} \quad [\text{expanding by } C_3] \\
 &= (1 + xyz)(y - x)(z - x)(z - y). \\
 \therefore \Delta = 0 &\Rightarrow (1 + xyz)(y - x)(z - x)(z - y) = 0 \\
 &\Rightarrow (1 + xyz) = 0 \quad [\because (y - x) \neq 0, (z - x) \neq 0, (z - y) \neq 0] \\
 &\Rightarrow xyz = -1.
 \end{aligned}$$

Hence, $xyz = -1$.

EXAMPLE 27 For any scalar p , prove that

$$\begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix} = (1 + pxyz)(x - y)(y - z)(z - x). \quad [\text{CBSE 2010}]$$

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix} \\
 &= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} \quad [\text{say } \Delta_1 + \Delta_2] \\
 &= (-1) \cdot \begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + p \cdot \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} \\
 &\quad [\text{by } C_2 \leftrightarrow C_3 \text{ in } \Delta_1] \quad [\text{taking } p \text{ common from } C_3 \text{ in } \Delta_2] \\
 &= (-1)(-1) \cdot \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + (pxyz) \cdot \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \\
 &\quad [\text{by } C_1 \leftrightarrow C_2] \quad [\text{taking } x, y, z \text{ common from } R_1, R_2, R_3 \text{ resp.}] \\
 &= (1 + pxyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \\
 &= (1 + pxyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y - x & y^2 - x^2 \\ 0 & z - x & z^2 - x^2 \end{vmatrix} \\
 &\quad [\text{by } R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1] \\
 &= (1 + pxyz)(y - x)(z - x) \cdot \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y + x \\ 0 & 1 & z + x \end{vmatrix} \\
 &\quad [\text{taking } (y - x) \text{ common from } R_2 \text{ and } (z - x) \text{ common from } R_3]
 \end{aligned}$$

$$\begin{aligned}
 &= (1 + pxyz)(y - x)(z - x) \cdot 1 \cdot \begin{vmatrix} 1 & y + x \\ 1 & z + x \end{vmatrix} \\
 &= (1 + pxyz)(y - x)(z - x) \cdot \{(z + y) - (y + x)\} \\
 &= (1 + pxyz)(y - x)(z - x)(z - y) \\
 &= (1 + pxyz)(x - y)(y - z)(z - x) = \text{RHS}.
 \end{aligned}$$

∴ LHS = RHS.

TYPE: SOME MORE PROBLEMS

EXAMPLE 28 Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = (abc) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1 \right) = (bc + ca + ab + abc).$$

[CBSE 2004, '08, '09, '12, '14]

SOLUTION The given determinant

$$\begin{aligned}
 &= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} \\
 &= (abc) \cdot \begin{vmatrix} \frac{1}{a} + 1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b} + 1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c} + 1 \end{vmatrix}
 \end{aligned}$$

[taking a, b, c common from R_1, R_2 and R_3 respectively]

$$= (abc) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1 \right) \cdot \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b} + 1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c} + 1 \end{vmatrix}$$

[applying $R_1 \rightarrow R_1 + R_2 + R_3$ and taking out $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1 \right)$ common from R_1]

$$= (abc) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1 \right) \cdot \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & \frac{1}{b} \\ -1 & -1 & \frac{1}{c} + 1 \end{vmatrix}$$

[applying $C_1 \rightarrow (C_1 - C_3)$ and $C_2 \rightarrow (C_2 - C_3)$]

$$= (abc) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1 \right) \cdot (1) \cdot \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} \quad [\text{expanding by 1st row}]$$

$$\begin{aligned}
 &= (abc) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1 \right) \cdot 1 \\
 &= (abc) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1 \right) = (bc + ca + ab + abc).
 \end{aligned}$$

Hence, the result follows.

EXAMPLE 29 Prove that

$$\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix} = 0.$$

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix} \\
 &= \frac{1}{abc} \cdot \begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & cba \\ 1 & c^3 & abc \end{vmatrix} \quad \begin{array}{l} \text{[multiplying } R_1, R_2, R_3 \text{ by } a, b, c \\ \text{respectively and dividing } \Delta \text{ by } abc] \end{array}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{abc} \cdot abc \cdot \begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix} \\
 &= (1 \times 0) = 0 \quad [\because C_1 \text{ and } C_3 \text{ are identical}].
 \end{aligned}$$

Hence, $\Delta = 0$.

EXAMPLE 30 Prove that

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = (1 + a^2 + b^2 + c^2).$$

[CBSE 2007, '08, '11C, '14]

SOLUTION Let the given determinant be Δ . Then,

$$\Delta = \begin{vmatrix} a\left(a + \frac{1}{a}\right) & ab & ac \\ ab & \left(b + \frac{1}{b}\right)b & bc \\ ca & cb & \left(c + \frac{1}{c}\right)c \end{vmatrix}$$

$$\begin{aligned}
 &= (abc) \cdot \begin{vmatrix} a + \frac{1}{a} & a & a \\ b & b + \frac{1}{b} & b \\ c & c & c + \frac{1}{c} \end{vmatrix} \\
 &\quad \text{[taking } a, b, c \text{ common from } C_1, C_2, C_3 \text{ respectively]} \\
 &= \frac{(abc)}{(abc)} \cdot \begin{vmatrix} a^2 + 1 & a^2 & a^2 \\ b^2 & b^2 + 1 & b^2 \\ c^2 & c^2 & c^2 + 1 \end{vmatrix} \\
 &\quad \text{[applying } R_1 \rightarrow aR_1, R_2 \rightarrow bR_2, R_3 \rightarrow cR_3 \\
 &\quad \text{and dividing the whole det. by } abc] \\
 &= \begin{vmatrix} 1 + a^2 + b^2 + c^2 & 1 + a^2 + b^2 + c^2 & 1 + a^2 + b^2 + c^2 \\ b^2 & b^2 + 1 & b^2 \\ c^2 & c^2 & c^2 + 1 \end{vmatrix} \\
 &\quad \text{[by } R_1 \rightarrow R_1 + R_2 + R_3] \\
 &= (1 + a^2 + b^2 + c^2) \cdot \begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2 + 1 & b^2 \\ c^2 & c^2 & c^2 + 1 \end{vmatrix} \\
 &\quad \text{[taking out } (1 + a^2 + b^2 + c^2) \text{ common from } R_1] \\
 &= (1 + a^2 + b^2 + c^2) \cdot \begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix} \quad \text{[by } C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1] \\
 &= (1 + a^2 + b^2 + c^2) \cdot 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \text{[expanded by } R_1] \\
 &= (1 + a^2 + b^2 + c^2) \cdot 1 \cdot (1 - 0) = (1 + a^2 + b^2 + c^2). \\
 &\text{Hence, } \Delta = (1 + a^2 + b^2 + c^2).
 \end{aligned}$$

EXAMPLE 31 Prove that $\begin{vmatrix} -bc & b^2 + bc & c^2 + bc \\ a^2 + ac & -ac & c^2 + ac \\ a^2 + ab & b^2 + ab & -ab \end{vmatrix} = (ab + bc + ac)^3$. [CBSE 2007]

SOLUTION We have

$$\begin{vmatrix} -bc & b^2 + bc & c^2 + bc \\ a^2 + ac & -ac & c^2 + ac \\ a^2 + ab & b^2 + ab & -ab \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} -abc & ab^2 + abc & ac^2 + abc \\ a^2b + abc & -abc & c^2b + abc \\ a^2c + abc & b^2c + abc & -abc \end{vmatrix}$$

[$R_1 \rightarrow aR_1, R_2 \rightarrow bR_2$ and $R_3 \rightarrow cR_3$ and dividing the det. by abc]

$$\begin{aligned}
&= \left(\frac{abc}{abc} \right) \begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix} \\
&\quad \text{[taking out } a, b, c \text{ common from } C_1, C_2, C_3 \text{ respectively]} \\
&= \begin{vmatrix} ab+bc+ac & ab+bc+ac & ab+bc+ac \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix} \quad [R_1 \rightarrow (R_1 + R_2 + R_3)] \\
&= (ab+bc+ac) \cdot \begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix} \\
&= (ab+bc+ac) \cdot \begin{vmatrix} 0 & 0 & 1 \\ 0 & -(ab+bc+ac) & bc+ab \\ (ab+bc+ac) & (ab+bc+ac) & -ab \end{vmatrix} \\
&\quad [C_1 \rightarrow (C_1 - C_3) \text{ and } C_2 \rightarrow (C_2 - C_3)] \\
&= (ab+bc+ac)^3 \quad \text{[expanding by } R_1].
\end{aligned}$$

EXAMPLE 32 Prove that

$$\begin{vmatrix} a^2 & a^2 - (b-c)^2 & bc \\ b^2 & b^2 - (c-a)^2 & ca \\ c^2 & c^2 - (a-b)^2 & ab \end{vmatrix} = (a^2 + b^2 + c^2)(a-b)(b-c)(c-a)(a+b+c).$$

[CBSE 2012C]

SOLUTION Let the value of the given determinant be Δ . Then,

$$\begin{aligned}
\Delta &= \begin{vmatrix} a^2 & a^2 - (b-c)^2 & bc \\ b^2 & b^2 - (c-a)^2 & ca \\ c^2 & c^2 - (a-b)^2 & ab \end{vmatrix} \\
&= \begin{vmatrix} a^2 & -(b-c)^2 & bc \\ b^2 & -(c-a)^2 & ca \\ c^2 & -(a-b)^2 & ab \end{vmatrix} \quad [\text{applying } C_2 \rightarrow (C_2 - C_1)] \\
&= (-1) \cdot \begin{vmatrix} a^2 & (b-c)^2 & bc \\ b^2 & (c-a)^2 & ca \\ c^2 & (a-b)^2 & ab \end{vmatrix} \\
&= (-1) \cdot \begin{vmatrix} a^2 & b^2 + c^2 - 2bc & bc \\ b^2 & c^2 + a^2 - 2ca & ca \\ c^2 & a^2 + b^2 - 2ab & ab \end{vmatrix} \\
&= (-1) \cdot \begin{vmatrix} a^2 & a^2 + b^2 + c^2 & bc \\ b^2 & a^2 + b^2 + c^2 & ca \\ c^2 & a^2 + b^2 + c^2 & ab \end{vmatrix} \quad [C_2 \rightarrow C_2 + C_1 + 2C_3]
\end{aligned}$$

$$\begin{aligned}
 &= (-1)(a^2 + b^2 + c^2) \cdot \begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix} \\
 &= (-1)(a^2 + b^2 + c^2) \cdot \begin{vmatrix} a^2 & 1 & bc \\ b^2 - a^2 & 0 & ca - bc \\ c^2 - a^2 & 0 & ab - bc \end{vmatrix} \\
 &= (-1)(a^2 + b^2 + c^2) \cdot \begin{vmatrix} a^2 & 1 & bc \\ b^2 - a^2 & 0 & (a-b)c \\ c^2 - a^2 & 0 & (a-c)b \end{vmatrix} \\
 &= (-1)(a^2 + b^2 + c^2)(b-a)(c-a) \cdot \begin{vmatrix} a^2 & 1 & bc \\ b+a & 0 & -c \\ c+a & 0 & -b \end{vmatrix}
 \end{aligned}$$

[taking out $(b-a)$ common from R_2 and $(c-a)$ common from R_3]

$$\begin{aligned}
 &= (-1)(a^2 + b^2 + c^2)(b-a)(c-a)(-1) \cdot \begin{vmatrix} b+a & -c \\ c+a & -b \end{vmatrix} \\
 &= (a^2 + b^2 + c^2)(b-a)(c-a) \cdot (-1) \cdot \begin{vmatrix} b+a & c \\ c+a & b \end{vmatrix} \\
 &= (a^2 + b^2 + c^2)(a-b)(c-a) \cdot \{b(b+a) - c(c+a)\} \\
 &= (a^2 + b^2 + c^2)(a-b)(c-a) \cdot \{(b^2 - c^2) + (ab - ac)\} \\
 &= (a^2 + b^2 + c^2)(a-b)(c-a) \cdot \{(b^2 - c^2) + a(b-c)\} \\
 &= (a^2 + b^2 + c^2)(a-b)(b-c)(c-a)(a+b+c).
 \end{aligned}$$

Hence, $\Delta = (a^2 + b^2 + c^2)(a-b)(b-c)(c-a)(a+b+c)$.

TYPE: DETERMINANTS ON TRIGONOMETRY

EXAMPLE 33 Prove that

$$\begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix} = 1.$$

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix} \\
 &= \frac{1}{\sin \alpha \cos \alpha} \cdot \begin{vmatrix} \sin \alpha \cos \alpha \cos \beta & \sin \alpha \cos \alpha \sin \beta & -\sin^2 \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \alpha \cos \beta & \sin \alpha \cos \alpha \sin \beta & \cos^2 \alpha \end{vmatrix}
 \end{aligned}$$

[$R_1 \rightarrow (\sin \alpha)R_1$, $R_3 \rightarrow (\cos \alpha)R_3$ and dividing Δ by $(\sin \alpha \cos \alpha)$]

$$\begin{aligned}
&= \frac{1}{\sin \alpha \cos \alpha} \cdot \begin{vmatrix} 0 & 0 & -1 \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \alpha \cos \beta & \sin \alpha \cos \alpha \sin \beta & \cos^2 \alpha \end{vmatrix} \\
&\quad [R_1 \rightarrow (R_1 - R_3)] \\
&= \frac{1}{\sin \alpha \cos \alpha} \cdot (-1) \cdot \begin{vmatrix} -\sin \beta & \cos \beta \\ \sin \alpha \cos \alpha \cos \beta & \sin \alpha \cos \alpha \sin \beta \end{vmatrix} \\
&= \frac{(\sin \alpha \cos \alpha)}{(\sin \alpha \cos \alpha)} \cdot (-1) \cdot \begin{vmatrix} -\sin \beta & \cos \beta \\ \cos \beta & \sin \beta \end{vmatrix} \\
&\quad [\text{taking } \sin \alpha \cos \alpha \text{ common from } R_2] \\
&= (-1) \cdot [-\sin^2 \beta - \cos^2 \beta] = (\sin^2 \beta + \cos^2 \beta) = 1.
\end{aligned}$$

Hence, $\Delta = 1$.

EXAMPLE 34 If $A + B + C = \pi$, prove that

$$\begin{vmatrix} \sin(A+B+C) & \sin(A+C) & \cos C \\ -\sin B & 0 & \tan C \\ \cos(A+B) & \tan(B+C) & 0 \end{vmatrix} = 0.$$

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
\Delta &= \begin{vmatrix} \sin(A+B+C) & \sin(A+C) & \cos C \\ -\sin B & 0 & \tan C \\ \cos(A+B) & \tan(B+C) & 0 \end{vmatrix} \\
&= \begin{vmatrix} \sin \pi & \sin(\pi-B) & \cos C \\ -\sin B & 0 & \tan C \\ \cos(\pi-C) & \tan(\pi-A) & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ -\cos C & -\tan A & 0 \end{vmatrix} \\
&\quad [\because \sin \pi = 0, \sin(\pi-B) = \sin B; \cos(\pi-C) = -\cos C \text{ and } \tan(\pi-A) = -\tan A] \\
&= \sin B \cdot \begin{vmatrix} \sin B & \cos C \\ -\tan A & 0 \end{vmatrix} - \cos C \cdot \begin{vmatrix} \sin B & \cos C \\ 0 & \tan A \end{vmatrix} \\
&\quad \{\text{expanded by } C_1\} \\
&= (\sin B \cdot \tan A \cos C) - (\sin B \tan A \cdot \cos C) = 0.
\end{aligned}$$

Hence, $\Delta = 0$.

EXAMPLE 35 If $A + B + C = \pi$, show that

$$\begin{vmatrix} \sin^2 A & \sin A \cos A & \cos^2 A \\ \sin^2 B & \sin B \cos B & \cos^2 B \\ \sin^2 C & \sin C \cos C & \cos^2 C \end{vmatrix} = -\sin(A-B) \sin(B-C) \sin(C-A).$$

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} 1 & (1/2) \sin 2A & (1/2)(1 + \cos 2A) \\ 1 & (1/2) \sin 2B & (1/2)(1 + \cos 2B) \\ 1 & (1/2) \sin 2C & (1/2)(1 + \cos 2C) \end{vmatrix} \quad [C_1 \rightarrow C_1 + C_3] \\
 &= \frac{1}{4} \begin{vmatrix} 1 & \sin 2A & 1 + \cos 2A \\ 1 & \sin 2B & 1 + \cos 2B \\ 1 & \sin 2C & 1 + \cos 2C \end{vmatrix} \\
 &= \frac{1}{4} \begin{vmatrix} 1 & \sin 2A & 1 + \cos 2A \\ 0 & \sin 2B - \sin 2A & \cos 2B - \cos 2A \\ 0 & \sin 2C - \sin 2A & \cos 2C - \cos 2A \end{vmatrix} \quad \left\{ \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \right\} \\
 &= \frac{1}{4} \begin{vmatrix} 1 & \sin 2A & 1 + \cos 2A \\ 0 & 2\cos(A+B)\sin(B-A) & 2\sin(A+B)\sin(A-B) \\ 0 & 2\cos(A+C)\sin(C-A) & 2\sin(A+C)\sin(A-C) \end{vmatrix} \\
 &= \sin(A-B)\sin(A-C) \cdot \begin{vmatrix} 1 & \sin 2A & 1 + \cos 2A \\ 0 & -\cos(A+B) & \sin(A+B) \\ 0 & -\cos(A+C) & \sin(A+C) \end{vmatrix} \\
 &\quad [\text{taking } 2\sin(A-B) \text{ common from } R_2 \text{ and } 2\sin(A-C) \text{ common from } R_3] \\
 &= \sin(A-B)\sin(A-C) \cdot \begin{vmatrix} 1 & \sin 2A & 1 + \cos 2A \\ 0 & -\cos(\pi - C) & \sin(\pi - C) \\ 0 & -\cos(\pi - B) & \sin(\pi - B) \end{vmatrix} \\
 &\quad [\because A + B + C = \pi] \\
 &= \sin(A-B)\sin(A-C) \cdot \begin{vmatrix} 1 & \sin 2A & 1 + \cos 2A \\ 0 & \cos C & \sin C \\ 0 & \cos B & \sin B \end{vmatrix} \\
 &= \sin(A-B)\sin(A-C) [\sin B \cos C - \cos B \sin C] \\
 &= \sin(A-B)\sin(A-C)\sin(B-C) \\
 &= -\sin(A-B)\sin(B-C)\sin(C-A).
 \end{aligned}$$

Hence, $\Delta = -\sin(A-B)\sin(B-C)\sin(C-A)$.

EXAMPLE 36 Prove that
$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix} = 0.$$

SOLUTION Let the given determinant be Δ .

Then, applying $C_3 \rightarrow C_3 + (\sin \delta) C_1 - (\cos \delta) C_2$, we get

$$\Delta = \begin{vmatrix} \sin \alpha & \cos \alpha & 0 \\ \sin \beta & \cos \beta & 0 \\ \sin \gamma & \cos \gamma & 0 \end{vmatrix} = 0 \quad [\because \text{each element in } C_3 \text{ is } 0].$$

TYPE: EQUALITY OF TWO DETERMINANTS

EXAMPLE 37 Without expanding prove that

$$\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2) \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}.$$

SOLUTION We have

$$\begin{aligned} \text{LHS} &= \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \\ &= \begin{vmatrix} a-ax^2 & c-cx^2 & p-px^2 \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \quad [\text{applying } R_1 \rightarrow R_1 - xR_2] \\ &= \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \\ &= (1-x^2) \begin{vmatrix} a & c & p \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \quad [\text{taking out } (1-x^2) \text{ common from } R_1] \\ &= (1-x^2) \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix} \quad [\text{applying } R_2 \rightarrow R_2 - xR_1] \\ &= \text{RHS.} \end{aligned}$$

Hence, LHS = RHS.

EXAMPLE 38 Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}. \quad [\text{CBSE 2001C}]$$

SOLUTION We have

$$\text{LHS} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

$$= \frac{1}{abc} \cdot \begin{vmatrix} a^2 & a^2 & abc \\ b^2 & b^2 & abc \\ c^2 & c^2 & abc \end{vmatrix}$$

$[R_1 \rightarrow aR_1, R_2 \rightarrow bR_2, R_3 \rightarrow cR_3 \text{ and dividing the whole det. by } abc]$

$$= \frac{1}{abc} \cdot abc \cdot \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \quad [\text{taking } abc \text{ common from } C_3]$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \quad [C_2 \leftrightarrow C_3 \text{ and } C_1 \leftrightarrow C_2]$$

= RHS.

Hence, LHS = RHS.

EXAMPLE 39 Using the properties of determinants, prove that

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}. \quad [\text{CBSE 2007, '10C}]$$

SOLUTION We have

$$\begin{aligned} \text{LHS} &= \begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} \\ &= \begin{vmatrix} -2c & b+c & c+a \\ -2a & c+a & a+b \\ -2b & a+b & b+c \end{vmatrix} \quad [\text{applying } C_1 \rightarrow C_1 - (C_2 + C_3)] \\ &= (-2) \begin{vmatrix} c & b+c & c+a \\ a & c+a & a+b \\ b & a+b & b+c \end{vmatrix} \quad [\text{taking out } (-2) \text{ common from } C_1] \\ &= (-2) \cdot \begin{vmatrix} c & b & a \\ a & c & b \\ b & a & c \end{vmatrix} \quad [\text{applying } C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1] \\ &= 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \quad [\text{applying } C_1 \leftrightarrow C_3] \end{aligned}$$

= RHS.

Hence, LHS = RHS.

EXAMPLE 40 Show that

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}. \quad [\text{CBSE 2008, '10C, '12C, '14}]$$

SOLUTION We have

$$\begin{aligned} \text{LHS} &= \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} \\ &= 2 \begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix} \\ &\quad [\text{applying } C_1 \rightarrow (C_1 + C_2 + C_3) \text{ and taking out 2 common from } C_1] \\ &= 2 \begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix} \quad [C_2 \rightarrow (C_2 - C_1), C_3 \rightarrow (C_3 - C_1)] \\ &= 2(-1)(-1) \cdot \begin{vmatrix} a+b+c & b & c \\ p+q+r & q & r \\ x+y+z & y & z \end{vmatrix} \\ &\quad [\text{taking out } (-1) \text{ common from each one of } C_2 \text{ and } C_3] \\ &= 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \text{RHS} \quad [\text{applying } C_1 \rightarrow C_1 - (C_2 + C_3)] \end{aligned}$$

Hence, LHS = RHS.

EXAMPLE 41 Show that $\Delta = \Delta_1$, where

$$\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} \text{ and } \Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}. \quad [\text{CBSE 2014C}]$$

SOLUTION We have

$$\begin{aligned} \Delta &= \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} \quad [\text{interchanging the rows and columns}] \end{aligned}$$

$$= (xyz) \begin{vmatrix} A & B & C \\ x & y & z \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \end{vmatrix}$$

[applying $C_1 \rightarrow \frac{1}{x}C_1, C_2 \rightarrow \frac{1}{y}C_2, C_3 \rightarrow \frac{1}{z}C_3$ and multiplying the whole determinant by xyz]

$$= \frac{(xyz)}{(xyz)} \cdot \begin{vmatrix} A & B & C \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

[applying $R_3 \rightarrow (xyz)R_3$ and dividing the whole determinant by xyz]

$$= \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix} = \Delta_1.$$

Hence, $\Delta = \Delta_1$.

EXAMPLE 42 If $\Delta_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}$ then without expanding Δ_1

and Δ_2 , prove that $\Delta_1 + \Delta_2 = 0$.

SOLUTION We have

$$\Delta_2 = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}$$

$$= (-1) \cdot \begin{vmatrix} q & -b & y \\ p & -a & x \\ r & -c & z \end{vmatrix} \quad [\text{taking } (-1) \text{ common from } R_2]$$

$$= (-1)(-1) \cdot \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix} \quad [\text{taking } (-1) \text{ common from } C_2]$$

$$= \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix} = \begin{vmatrix} q & p & r \\ b & a & c \\ y & x & z \end{vmatrix} \quad [\text{interchanging rows and columns}]$$

$$= (-1) \begin{vmatrix} p & q & r \\ a & b & c \\ x & y & z \end{vmatrix} \quad [\text{applying } C_1 \leftrightarrow C_2]$$

$$\begin{aligned}
 &= (-1)(-1) \cdot \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \quad [\text{applying } R_1 \leftrightarrow R_2] \\
 &= \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \\
 &= (-1) \cdot \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = -\Delta_1 \quad [\text{applying } R_2 \leftrightarrow R_3].
 \end{aligned}$$

Thus, $\Delta_2 = -\Delta_1$ and hence $\Delta_1 + \Delta_2 = 0$.

TYPE: EQUATIONS OF DETERMINANTS

EXAMPLE 43 Solve for x :

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0. \quad [\text{CBSE 2004, '05, '11}]$$

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}
 \Delta &= \begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} \\
 &= \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} \quad [C_1 \rightarrow (C_1 + C_2 + C_3)] \\
 &= (3a-x) \cdot \begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} \quad [\text{taking } (3a-x) \text{ common from } C_1] \\
 &= (3a-x) \cdot \begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} \quad [R_2 \rightarrow (R_2 - R_1) \text{ and } R_3 \rightarrow (R_3 - R_1)] \\
 &= (3a-x) \cdot 1 \cdot \begin{vmatrix} 2x & 0 \\ 0 & 2x \end{vmatrix} \quad [\text{expanding by } C_1] \\
 &= 4(3a-x)x^2. \\
 \therefore \Delta = 0 &\Leftrightarrow 4(3a-x)x^2 = 0 \\
 &\Leftrightarrow x = 0 \quad \text{or} \quad x = 3a.
 \end{aligned}$$

Hence, solution set = $\{0, 3a\}$.

EXAMPLE 44 Solve $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$ [CBSE 2011]

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}\Delta &= \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} \\ &= \begin{vmatrix} x-2 & 1 & 2 \\ x-4 & -1 & -4 \\ x-8 & -11 & -40 \end{vmatrix} \quad [C_2 \rightarrow (C_2 - 2C_1), C_3 \rightarrow (C_3 - 3C_1)] \\ &= \begin{vmatrix} x-2 & 1 & 2 \\ -2 & -2 & -6 \\ -6 & -12 & -42 \end{vmatrix} \quad [R_2 \rightarrow (R_2 - R_1), R_3 \rightarrow (R_3 - R_1)] \\ &= (-2) \cdot (-6) \cdot \begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 7 \end{vmatrix} \\ &= 12 \cdot \begin{vmatrix} x-3 & 1 & 2 \\ 0 & 1 & 3 \\ -1 & 2 & 7 \end{vmatrix} \quad [C_1 \rightarrow (C_1 - C_2)] \\ &= 12 \cdot [(x-3)(7-6) - 1 \cdot (3-2)] \\ &= 12 \cdot (x-4).\end{aligned}$$

$$\therefore \Delta = 0 \Leftrightarrow 12(x-4) = 0 \Leftrightarrow x = 4.$$

Hence, solution set = {4}.

EXAMPLE 45 If $a + b + c = 0$ and $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ then show that

$$x = 0 \text{ or } x = \sqrt{\left(\frac{3}{2}\right)(a^2 + b^2 + c^2)}.$$

SOLUTION Let the given determinant be Δ . Then,

$$\begin{aligned}\Delta &= \begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix} \quad [C_1 \rightarrow (C_1 + C_2 + C_3)] \\ &= (a+b+c-x) \begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} \\ &= (a+b+c-x) \begin{vmatrix} 1 & c & b \\ 0 & b-x-c & a-b \\ 0 & a-c & c-x-b \end{vmatrix} \\ &\quad [R_2 \rightarrow (R_2 - R_1) \text{ and } R_3 \rightarrow (R_3 - R_1)]\end{aligned}$$

$$= (a + b + c - x)[(b - x - c)(c - x - b) - (a - c)(a - b)]$$

$$= (a + b + c - x)[x^2 - (a^2 + b^2 + c^2 - ab - bc - ca)].$$

$$\text{Now, } \Delta = 0 \Leftrightarrow (a + b + c - x)[x^2 - (a^2 + b^2 + c^2 - ab - bc - ca)] = 0$$

$$\Leftrightarrow x = a + b + c \text{ or } x = \pm \sqrt{(a^2 + b^2 + c^2 - ab - bc - ca)}$$

$$\Leftrightarrow x = 0 \text{ or } x = \pm \sqrt{(3/2)(a^2 + b^2 + c^2)}$$

$$[\because a + b + c = 0 \Leftrightarrow (a^2 + b^2 + c^2) = -2(ab + bc + ca)]$$

$$\Leftrightarrow (ab + bc + ca) = -(1/2)(a^2 + b^2 + c^2)].$$

EXERCISE 6A

Very-Short-Answer Questions

1. If A is a 2×2 matrix such that $|A| \neq 0$ and $|A| = 5$, write the value of $|4A|$.
2. If A is a 3×3 matrix such that $|A| \neq 0$ and $|3A| = k|A|$ then write the value of k . [CBSE 2014]
3. Let A be a square matrix of order 3, write the value of $|2A|$, where $|A| = 4$. [CBSE 2012]
4. If A_{ij} is the cofactor of the element a_{ij} of $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ then write the value of $(a_{32}A_{32})$. [CBSE 2013]
5. Evaluate $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$. [CBSE 2008]
6. Evaluate $\begin{vmatrix} a + ib & c + id \\ -c + id & a - ib \end{vmatrix}$. [CBSE 2008]
7. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x . [CBSE 2014]
8. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x . [CBSE 2014]
9. If $\begin{vmatrix} 2x & x + 3 \\ 2(x + 1) & x + 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix}$, find the value of x . [CBSE 2013C]
10. If $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$, find the value of $3|A|$. [CBSE 2011C]
11. Evaluate $2 \begin{vmatrix} 7 & -2 \\ -10 & 5 \end{vmatrix}$. [CBSE 2009C]

12. Evaluate $\begin{vmatrix} \sqrt{6} & \sqrt{5} \\ \sqrt{20} & \sqrt{24} \end{vmatrix}$. [CBSE 2009C]

13. Evaluate $\begin{vmatrix} 2\cos \theta & -2\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix}$. [CBSE 2008]

14. Evaluate $\begin{vmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{vmatrix}$.

15. Evaluate $\begin{vmatrix} \sin 60^\circ & \cos 60^\circ \\ -\sin 30^\circ & \cos 30^\circ \end{vmatrix}$.

16. Evaluate $\begin{vmatrix} \cos 65^\circ & \sin 65^\circ \\ \sin 25^\circ & \cos 25^\circ \end{vmatrix}$.

17. Evaluate $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$. [CBSE 2011]

18. Evaluate $\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$.

19. Without expanding the determinant, prove that $\begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix} = 0$.

SINGULAR MATRIX A square matrix A is said to be singular if $|A| = 0$.

Also, A is called **nonsingular** if $|A| \neq 0$.

20. For what value of x , the given matrix $A = \begin{bmatrix} 3-2x & x+1 \\ 2 & 4 \end{bmatrix}$ is a singular matrix? [CBSE 2013C]

21. Evaluate $\begin{vmatrix} 14 & 9 \\ -8 & -7 \end{vmatrix}$.

22. Evaluate $\begin{vmatrix} \sqrt{3} & \sqrt{5} \\ -\sqrt{5} & 3\sqrt{3} \end{vmatrix}$.

ANSWERS (EXERCISE 6A)

1. 80 2. $k = 27$ 3. 32 4. 110 5. $x^3 - x^2 + 2$ 6. $(a^2 + b^2 + c^2 + d^2)$
 7. $x = -2$ 8. $x = \pm 6$ 9. $x = 1$ 10. 6 11. 30 12. 2 13. 2 14. 1
 15. 1 16. 0 17. 0 18. 8 20. $x = 1$ 21. -26 22. 14

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 6A)

1. Since A is a 2×2 matrix, so $|4A| = (4 \times 4) \cdot |A|$.
2. Since A is a 3×3 matrix, so $|3A| = (3 \times 3 \times 3) \cdot |A|$.
4. $M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = (8 - 30) = -22$.
So, $A_{32} = (-1)^{3+2} \cdot M_{32} = (-1)(-22) = 22$.
 $\therefore (a_{32}A_{32}) = (5 \times 22) = 110$.
5. $\Delta = (x^2 - x + 1)(x + 1) - (x + 1)(x - 1) = (x^3 + 1) - (x^2 - 1) = (x^3 - x^2 + 2)$.
6. $\Delta = (a + ib)(a - ib) - (c + id)(-c + id)$
 $= \{a^2 - (ib)^2\} - \{(id)^2 - c^2\} = (a^2 + b^2) - (-d^2 - c^2) = (a^2 + b^2 + c^2 + d^2)$.
7. $(3x \times 4) - 7 \times (-2) = 32 - 42 \Rightarrow 12x + 14 = -10 \Rightarrow 12x = -24 \Rightarrow x = -2$.
8. $2x^2 - 40 = 18 + 14 \Rightarrow 2x^2 = 32 + 40 = 72 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$.
9. $2x(x + 1) - 2(x + 1)(x + 3) = 3 - 15$
 $\Rightarrow 2(x + 1)\{x - (x + 3)\} = -12 \Rightarrow -6(x + 1) = -12 \Rightarrow x + 1 = 2 \Rightarrow x = 1$.
10. $3|A| = 3 \cdot \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3 \cdot (6 - 4) = 3 \times 2 = 6$.
11. $2\Delta = 2(35 - 20) = (2 \times 15) = 30$.
12. $\Delta = (\sqrt{6} \times \sqrt{24}) - (\sqrt{20} \times \sqrt{5}) = \{\sqrt{6 \times 24} - \sqrt{20 \times 5}\}$
 $= \{\sqrt{144} - \sqrt{100}\} = (12 - 10) = 2$.
13. $\Delta = (2 \cos^2 \theta + 2 \sin^2 \theta) = 2(\cos^2 \theta + \sin^2 \theta) = (2 \times 1) = 2$.
14. $\Delta = (\cos^2 \alpha + \sin^2 \alpha) = 1$.
15. $\Delta = (\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ) = \sin(60^\circ + 30^\circ) = \sin 90^\circ = 1$.
16. $\Delta = \{\cos 65^\circ \cos 25^\circ - \sin 65^\circ \sin 25^\circ\} = \cos(65^\circ + 25^\circ) = \cos 90^\circ = 0$.
18. $\Delta = (-2) \cdot \begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix} = (-2)(12 - 16) = (-2) \times (-4) = 8$.
19. Applying $C_1 \rightarrow C_1 - 8C_3$, we get
 $\Delta = \begin{vmatrix} 1 & 1 & 5 \\ 7 & 7 & 9 \\ 5 & 5 & 3 \end{vmatrix} = 0$ [$\because C_1$ and C_2 are identical].
20. A is singular $\Leftrightarrow |A| = 0$
 $\Leftrightarrow \begin{vmatrix} 3 - 2x & x + 1 \\ 2 & 4 \end{vmatrix} = 0$
 $\Leftrightarrow 4(3 - 2x) - 2(x + 1) = 0 \Leftrightarrow 10x = 10 \Leftrightarrow x = 1$.

EXERCISE 6B**Evaluate:**

$$1. \begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

$$2. \begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix}$$

$$3. \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$4. \begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$$

Using properties of determinants prove that:

$$5. \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a).$$

[CBSE 2006]

$$6. \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

$$7. \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1.$$

[CBSE 2009]

$$8. \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z).$$

[CBSE 2014]

$$9. \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x+2a)(x-a)^2.$$

$$10. \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(x-4)^2.$$

[CBSE 2009, '11]

$$11. \begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2.$$

[CBSE 2014]

$$12. \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3.$$

$$13. \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$$

[CBSE 2013]

$$14. \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3(x+y+z)(xy+yz+zx).$$

[CBSE 2013]

$$15. \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x). \quad [\text{CBSE 2010C, '11}]$$

$$16. \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3. \quad [\text{CBSE 2006, '12C}]$$

$$17. \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc. \quad [\text{CBSE 2006, '12, '14C}]$$

$$18. \begin{vmatrix} a & a+2b & a+2b+3c \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix} = -a^3.$$

$$19. \begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a).$$

$$20. \begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2).$$

$$21. \begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = -4(a-b)(b-c)(c-a).$$

$$22. \begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix} = -8.$$

$$23. \begin{vmatrix} (m+n)^2 & l^2 & mn \\ (n+l)^2 & m^2 & ln \\ (l+m)^2 & n^2 & lm \end{vmatrix} = (l^2+m^2+n^2)(l-m)(m-n)(n-l).$$

$$24. \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c).$$

$$25. \begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$$

$$26. \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3. \quad [\text{CBSE 2008, '09, '10C}]$$

$$27. \begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2).$$

$$28. \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0.$$

$$29. \begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3. \quad [\text{CBSE 2010}]$$

$$30. \begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix} = 0.$$

$$31. \begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2+c^2) \end{vmatrix} = (abc)(a^2+b^2+c^2)^3.$$

$$32. \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 0, \text{ where } \alpha, \beta, \gamma \text{ are in AP.} \quad [\text{CBSE 2007}]$$

$$33. \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2.$$

$$34. \text{ If } x \neq y \neq z \text{ and } \begin{vmatrix} x & x^3 & x^4-1 \\ y & y^3 & y^4-1 \\ z & z^3 & z^4-1 \end{vmatrix} = 0, \text{ prove that}$$

$$xyz(xy + yz + zx) = (x + y + z).$$

35. Prove that

$$\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2+b^2+c^2).$$

Without expanding the determinant, prove that:

$$36. \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad 37. \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix}$$

$$38. \text{ Show that } x = 2 \text{ is a root of the equation } \begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & 2+x \end{vmatrix} = 0. \quad [\text{CBSE 2007}]$$

Solve the following equations:

$$39. \begin{vmatrix} 1 & x & x^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0 \quad 40. \begin{vmatrix} x+a & b & c \\ a & x+b & c \\ b & b & x+c \end{vmatrix} = 0$$

$$41. \begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0 \quad 42. \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$$

$$43. \begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0 \quad 44. \begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$$

45. Prove that

$$\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2). \quad [\text{CBSE 2012C}]$$

ANSWERS (EXERCISE 6B)

1. -43 2. 132 3. 0 4. -8 39. $x = b$ or $x = c$ or $x = -(b+c)$
 40. $x = 0$ or $x = -(a+b+c)$ 41. $x = \frac{2}{3}$ or $x = \frac{11}{3}$ 42. $x = 1$ or $x = -9$
 43. $x = -9$ or $x = 2$ or $x = 7$ 44. $x = 1$ or $x = 2$ or $x = -3$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 6B)

1. Apply $C_1 \rightarrow C_1 - 3C_2, C_3 \rightarrow C_3 - C_2$.
Now, apply $C_2 \rightarrow C_2 - 13C_3$. Expand by R_2 .
2. Apply $C_1 \rightarrow C_1 - C_2$ and $C_3 \rightarrow C_3 - C_2$.
Take 3 common from C_1 and -4 common from C_3 .
Apply $C_2 \rightarrow C_2 - 26C_3$ and $C_3 \rightarrow C_3 - C_1$.
3. Apply $R_1 \rightarrow R_1 - 6R_3$.
4. Apply $C_3 \rightarrow C_3 - C_2$ and $C_2 \rightarrow C_2 - C_1$. Now, apply $C_3 \rightarrow C_3 - C_2$.
5. Apply $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$.

6. Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - 3R_1$.
7. Apply $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$.
8. Apply $C_1 \rightarrow C_1 + C_2 + C_3$ and take $(a + x + y + z)$ common from C_1 .
9. Apply $C_1 \rightarrow C_1 + C_2 + C_3$ and take $(x + 2a)$ common from C_1 .
10. Apply $R_1 \rightarrow R_1 + R_2 + R_3$ and take $(5x + 4)$ common from R_1 .
11. Apply $R_1 \rightarrow R_1 + R_2 + R_3$ and take $(5x + \lambda)$ common from R_1 .
12. Apply $R_1 \rightarrow R_1 - R_3, R_2 \rightarrow R_2 - R_3$ and then expand along C_3 .
13. Apply $R_1 \rightarrow R_1 + R_2 + R_3$ and take $3(x + y)$ common from R_1 .
14. Apply $C_1 \rightarrow C_1 + C_2 + C_3$ and take $(x + y + z)$ common from C_1 .
15. Take x, y and z common from C_1, C_2, C_3 respectively.
16. Apply $C_1 \rightarrow C_1 + C_3$ and $C_2 \rightarrow C_2 - C_3$.
17. Apply $R_1 \rightarrow R_1 - (R_2 + R_3)$. Take (-2) common from R_1 .
Now, apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$.
18. Apply $R_2 \rightarrow R_2 - 3R_1$ and $R_3 \rightarrow R_3 - 6R_1$.
19. Apply $R_2 \rightarrow R_2 + R_1$ and $R_3 \rightarrow R_3 + R_1$.
20. Apply $R_3 \rightarrow (R_3 - xR_1 - yR_2)$.
21. Apply $R_2 \rightarrow (R_2 - R_3)$ and take 4 common from R_2 .
Now, apply $R_3 \rightarrow R_3 - R_1 + 2R_2$.
22. Apply $R_1 \rightarrow R_1 - R_3$ and $R_2 \rightarrow R_2 - R_3$. Then, apply $R_2 \rightarrow R_2 + 2R_1$.
Now, $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$.
23. Apply $C_1 \rightarrow (C_1 + C_2 - 2C_3)$ and take $(l^2 + m^2 + n^2)$ common from C_1 .
24. Apply $C_1 \rightarrow (C_1 + C_2 - 2C_3)$.
25. Apply $R_1 \rightarrow R_1 - (R_2 + R_3)$ and expand by C_1 .
26. Apply $C_1 \rightarrow C_1 - bC_3$ and $C_2 \rightarrow C_2 + aC_3$.
Now, take $(1 + a^2 + b^2)$ common from each of C_1 and C_2 .
27. Apply $C_1 \rightarrow aC_1, C_2 \rightarrow bC_2, C_3 \rightarrow cC_3$ and divide the whole determinant by abc .
Now, apply $C_1 \rightarrow (C_1 + C_2 + C_3)$.
28. Taking bc, ca and ab common from R_1, R_2 and R_3 respectively, we get

$$\Delta = (a^2b^2c^2) \cdot \begin{vmatrix} bc & 1 & \frac{1}{b} + \frac{1}{c} \\ ca & 1 & \frac{1}{c} + \frac{1}{a} \\ ab & 1 & \frac{1}{a} + \frac{1}{b} \end{vmatrix}.$$

Now, $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$.

29. Apply $R_1 \rightarrow aR_1, R_2 \rightarrow bR_2, R_3 \rightarrow cR_3$ and divide Δ by abc .
Take a, b, c common from C_1, C_2 and C_3 respectively.

$$\begin{aligned} 30. \Delta &= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix} \\ &= (b-a)^2 \cdot \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix} \quad [\text{taking } (b-a) \text{ common from each of } C_1 \text{ and } C_3]. \end{aligned}$$

Now, apply $C_2 \rightarrow (C_2 - C_1 + C_3)$.

31. Take a, b, c common from C_1, C_2 and C_3 respectively.

Now, apply $C_1 \rightarrow (C_1 + C_2 + C_3)$.

32. Apply $R_1 \rightarrow (R_1 + R_2 + R_3)$ and use $\alpha + \gamma = 2\beta$.

Then, R_1 and R_2 are proportional and hence $\Delta = 0$.

$$33. \Delta = \begin{vmatrix} a^2 + 3a + 2 & a + 2 & 1 \\ a^2 + 5a + 6 & a + 3 & 1 \\ a^2 + 7a + 12 & a + 4 & 1 \end{vmatrix}.$$

Now, apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$.

$$35. \Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} 1 & bc & a^3 \\ 1 & ca & b^3 \\ 1 & ab & c^3 \end{vmatrix}.$$

36. Applying $R_1 \rightarrow aR_1, R_2 \rightarrow bR_2$ and $R_3 \rightarrow cR_3$, we get

$$\text{LHS} = \frac{1}{abc} \cdot \begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

37. Applying $C_3 \rightarrow C_3 - (a + b + c)C_1$, we get

$$\begin{aligned} \text{RHS} &= \begin{vmatrix} 1 & bc & -a \\ 1 & ca & -b \\ 1 & ab & -c \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \\ &= \frac{1}{abc} \cdot \begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \text{LHS}. \end{aligned}$$

38. Apply $R_1 \rightarrow (R_1 - R_2)$ and take $(x - 2)$ common from R_1 .

39. Apply $R_1 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_3 - R_2$.

Take $(x - b)$ common from R_1 .

40. Apply $C_1 \rightarrow C_1 + C_2 + C_3$ and take $(x + a + b + c)$ common from C_1 .

41. Apply $C_1 \rightarrow C_1 + C_2 + C_3$ and take $(3x - 2)$ common from C_1 .

42. Apply $C_1 \rightarrow C_1 + C_2 + C_3$ and take $(x + 9)$ common from C_1 .

43. Apply $R_1 \rightarrow R_1 + R_2 + R_3$ and take $(x + 9)$ common from R_1 .

44. Apply $R_1 \rightarrow R_1 - R_2$ and take $(x - 2)$ common from R_1 .

45. Apply $C_1 \rightarrow aC_1$ and divide Δ by a .

$$\begin{aligned} \therefore \Delta &= \frac{1}{a} \cdot \begin{vmatrix} a^2 & b - c & c + b \\ a^2 + ac & b & c - a \\ a^2 - ab & b + a & c \end{vmatrix} \\ &= \frac{1}{a} \begin{vmatrix} a^2 + b^2 + c^2 & b - c & c + b \\ a^2 + b^2 + c^2 & b & c - a \\ a^2 + b^2 + c^2 & b + a & c \end{vmatrix} \quad [\text{applying } C_1 \rightarrow C_1 + bC_2 + cC_3] \end{aligned}$$

Applications of Determinants

AREA OF A TRIANGLE IN DETERMINANT FORM We know that the area of a triangle whose vertices are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) is given by

$$\begin{aligned}\Delta &= \frac{1}{2}[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)] \\ &= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad (\text{in determinant form}).\end{aligned}$$

REMARK 1. Since area is a positive quantity, we always take the absolute value of the above determinant for the area.

REMARK 2. If three points A, B, C are collinear then $\text{ar}(\Delta ABC) = 0$.

CONDITION FOR COLLINEARITY OF THREE POINTS

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be three given points.

Then, A, B, C are collinear

$$\Leftrightarrow \text{ar}(\Delta ABC) = 0$$

$$\Leftrightarrow \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

$$\therefore A, B, C \text{ are collinear} \Leftrightarrow x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

SOLVED EXAMPLES

EXAMPLE 1 Find the area of a triangle whose vertices are $A(-2, -3)$, $B(3, 2)$ and $C(-1, -8)$.

SOLUTION Here, $(x_1 = -2, y_1 = -3)$, $(x_2 = 3, y_2 = 2)$ and $(x_3 = -1, y_3 = -8)$.

$$\begin{aligned}\therefore \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} &= \frac{1}{2} \begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix} \\ &= \frac{1}{2} \begin{vmatrix} -2 & -3 & 1 \\ 5 & 5 & 0 \\ 1 & -5 & 0 \end{vmatrix}\end{aligned}$$

$$[R_2 \rightarrow (R_2 - R_1) \text{ and } R_3 \rightarrow (R_3 - R_1)]$$

$$= \frac{1}{2} \cdot 1 \cdot \begin{vmatrix} 5 & 5 \\ 1 & -5 \end{vmatrix} = \frac{1}{2} \times (-25 - 5) = -15.$$

Hence, $\text{ar}(\Delta ABC) = |-15| = 15$ square units.

EXAMPLE 2 Show that the points $A(a, b+c)$, $B(b, c+a)$ and $C(c, a+b)$ are collinear.

SOLUTION Points A, B, C are collinear $\Leftrightarrow \text{ar}(\triangle ABC) = 0$.

$$\begin{aligned}
 \text{Now, } \text{ar}(\triangle ABC) &= \frac{1}{2} \cdot \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} \\
 &= \frac{1}{2} \cdot \begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix} \quad [C_2 \rightarrow (C_2 + C_1)] \\
 &= \frac{1}{2}(a+b+c) \cdot \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} \\
 &= \frac{1}{2}(a+b+c) \times 0 = 0 \quad [\because C_2 \text{ and } C_3 \text{ are identical}].
 \end{aligned}$$

Hence, the given points are collinear.

EXAMPLE 3 If the points (a, b) , (a', b') and $(a-a', b-b')$ are collinear, show that $ab' = a'b$.

SOLUTION The given points are collinear

$$\begin{aligned}
 &\Leftrightarrow \begin{vmatrix} a & b & 1 \\ a' & b' & 1 \\ a-a' & b-b' & 1 \end{vmatrix} = 0 \\
 &\Leftrightarrow \begin{vmatrix} a & b & 1 \\ a'-a & b'-b & 0 \\ -a' & -b' & 0 \end{vmatrix} = 0 \quad [R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1] \\
 &\Leftrightarrow 1 \cdot [(a' - a)(-b') + a'(b' - b)] = 0 \quad [\text{expanding by } C_3] \\
 &\Leftrightarrow ab' - a'b = 0 \\
 &\Leftrightarrow ab' = a'b.
 \end{aligned}$$

EXAMPLE 4 Find the value of k in order that the points $(5, 5)$, $(k, 1)$ and $(10, 7)$ are collinear.

SOLUTION The given points are collinear

$$\begin{aligned}
 &\Leftrightarrow \begin{vmatrix} 5 & 5 & 1 \\ k & 1 & 1 \\ 10 & 7 & 1 \end{vmatrix} = 0 \\
 &\Leftrightarrow \begin{vmatrix} 5 & 5 & 1 \\ k-5 & -4 & 0 \\ 5 & 2 & 0 \end{vmatrix} = 0 \quad [R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]
 \end{aligned}$$

$$\Leftrightarrow 1 \cdot [2(k-5) + 20] = 0$$

$$\Leftrightarrow 2k + 10 = 0 \Leftrightarrow k = -5.$$

Hence, $k = -5$.

EXAMPLE 5 Let $A(1, 3)$, $B(0, 0)$ and $C(k, 0)$ be three points such that $\text{ar}(\triangle ABC) = 3$ sq units. Find the value of k .

SOLUTION We have

$$\text{ar}(\triangle ABC) = 3 \text{ sq units}$$

$$\Leftrightarrow \frac{1}{2} \cdot \begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3 \Leftrightarrow \begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 6$$

$$\Leftrightarrow (-1) \cdot \begin{vmatrix} 1 & 3 \\ k & 0 \end{vmatrix} = \pm 6 \Leftrightarrow 3k = \pm 6 \Leftrightarrow k = \pm 2$$

Hence, $k = \pm 2$.

EXAMPLE 6 Find the equation of the line joining the points $A(1, 2)$ and $B(3, 6)$, using determinants.

SOLUTION Let $P(x, y)$ be a point on AB .

Then, the points A , P and B are collinear.

$$\therefore \text{ar}(\triangle APB) = 0$$

$$\Rightarrow \frac{1}{2} \cdot \begin{vmatrix} 1 & 2 & 1 \\ x & y & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 2 & 1 \\ x & y & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & 1 \\ x & y & 1 \\ 0 & 0 & -2 \end{vmatrix} = 0 \quad [R_3 \rightarrow R_3 - 3R_1]$$

$$\Rightarrow (-2) \cdot \begin{vmatrix} 1 & 2 \\ x & y \end{vmatrix} = 0 \Rightarrow (y - 2x) = 0 \Rightarrow y = 2x.$$

Hence, the required equation is $y = 2x$.

EXERCISE 6C

1. Find the area of the triangle whose vertices are:

- (i) $A(3, 8)$, $B(-4, 2)$ and $C(5, -1)$
- (ii) $A(-2, 4)$, $B(2, -6)$ and $C(5, 4)$
- (iii) $A(-8, -2)$, $B(-4, -6)$ and $C(-1, 5)$
- (iv) $P(0, 0)$, $Q(6, 0)$ and $R(4, 3)$
- (v) $P(1, 1)$, $Q(2, 7)$ and $R(10, 8)$

[CBSE 2007]

2. Use determinants to show that the following points are collinear.

- (i) $A(2, 3)$, $B(-1, -2)$ and $C(5, 8)$

- (ii) $A(3, 8)$, $B(-4, 2)$ and $C(10, 14)$
 (iii) $P(-2, 5)$, $Q(-6, -7)$ and $R(-5, -4)$
- Find the value of k for which the points $A(3, -2)$, $B(k, 2)$ and $C(8, 8)$ are collinear.
 - Find the value of k for which the points $P(5, 5)$, $Q(k, 1)$ and $R(11, 7)$ are collinear.
 - Find the value of k for which the points $A(1, -1)$, $B(2, k)$ and $C(4, 5)$ are collinear.
 - Find the value of k for which the area of $\triangle ABC$ having vertices $A(2, -6)$, $B(5, 4)$ and $C(k, 4)$ is 35 sq units.
 - If $A(-2, 0)$, $B(0, 4)$ and $C(0, k)$ be three points such that area of $\triangle ABC$ is 4 sq units, find the value of k .
 - If the points $A(a, 0)$, $B(0, b)$ and $C(1, 1)$ are collinear, prove that $\frac{1}{a} + \frac{1}{b} = 1$.

ANSWERS (EXERCISE 6C)

- (i) 37.5 sq units (ii) 35 sq units (iii) 28 sq units (iv) 9 sq units
 (v) 23.5 sq units
- $k = 5$ 4. $k = -7$ 5. $k = 1$ 6. $k = -2$ or $k = 12$ 7. $k = 0$ or $k = 8$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

- $\begin{vmatrix} \cos 70^\circ & \sin 20^\circ \\ \sin 70^\circ & \cos 20^\circ \end{vmatrix} = ?$
 (a) 1 (b) 0 (c) $\cos 50^\circ$ (d) $\sin 50^\circ$
- $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 15^\circ & \cos 15^\circ \end{vmatrix} = ?$
 (a) 1 (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) none of these
- $\begin{vmatrix} \sin 23^\circ & -\sin 7^\circ \\ \cos 23^\circ & \cos 7^\circ \end{vmatrix} = ?$
 (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $\sin 16^\circ$ (d) $\cos 16^\circ$
- $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} = ?$
 (a) $(a^2 + b^2 - c^2 - d^2)$ (b) $(a^2 - b^2 + c^2 - d^2)$
 (c) $(a^2 + b^2 + c^2 + d^2)$ (d) none of these

5. If ω is a complex root of unity then $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = ?$

- (a) 1 (b) -1 (c) 0 (d) none of these

6. If ω is a complex cube root of unity then the value of $\begin{vmatrix} 1 & \omega & 1 + \omega \\ 1 + \omega & 1 & \omega \\ \omega & 1 + \omega & 1 \end{vmatrix}$ is

- (a) 2 (b) 4 (c) 0 (d) -3

7. $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix} = ?$

- (a) 8 (b) -8 (c) 16 (d) 142

8. $\begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix} = ?$

- (a) 2 (b) 6 (c) 24 (d) 120

9. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = ?$

[CBSE 2009]

- (a) $(a + b + c)$ (b) $3(a + b + c)$ (c) $3abc$ (d) 0

10. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = ?$

[CBSE 2009]

- (a) 0 (b) 1 (c) -1 (d) none of these

11. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = ?$

[CBSE 2009]

- (a) $(a-b)(b-c)(c-a)$ (b) $-(a-b)(b-c)(c-a)$
(c) $(a-b)(b-c)(c-a)(a+b+c)$ (d) $abc(a-b)(b-c)(c-a)$

12. $\begin{vmatrix} \sin \alpha & \cos \alpha & \sin (\alpha + \delta) \\ \sin \beta & \cos \beta & \sin (\beta + \delta) \\ \sin \gamma & \cos \gamma & \sin (\gamma + \delta) \end{vmatrix} = ?$

- (a) 0
(b) 1
(c) $\sin (\alpha + \delta) + \sin (\beta + \delta) + \sin (\gamma + \delta)$
(d) none of these

13. If a, b, c be distinct positive real numbers then the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is

(a) positive

(b) negative

(c) a perfect square

(d) 0

14. $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = ?$

(a) 0

(b) x^3

(c) y^3

(d) none of these

15. $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = ?$

(a) $(a-1)$

(b) $(a-1)^2$

(c) $(a-1)^3$

(d) none of these

16. $\begin{vmatrix} a & a+2b & a+2b+3c \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix} = ?$

(a) a^3

(b) $-a^3$

(c) 0

(d) none of these

17. $\begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix} = ?$

(a) $(a+b+c)(a-c)$

(b) $(a+b+c)(b-c)$

(c) $(a+b+c)(a-c)^2$

(d) $(a+b+c)(b-c)^2$

18. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = ?$

(a) $(x+y)$

(b) $(x-y)$

(c) xy

(d) none of these

19. $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = ?$

(a) $(a-b)(b-c)(c-a)$

(b) $-(a-b)(b-c)(c-a)$

(c) $(a+b)(b+c)(c+a)$

(d) none of these

20. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = ?$

(a) $4abc$

(b) $2(a+b+c)$

(c) $(ab+bc+ca)$

(d) none of these

21. $\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix} = ?$

- (a) $a + b + c$ (b) $2(a + b + c)$ (c) $4abc$ (d) $a^2b^2c^2$

22. $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix} = ?$

- (a) -2 (b) 2 (c) $x^2 - 2$ (d) $x^2 + 2$

23. If $\begin{vmatrix} 5 & 3 & -1 \\ -7 & x & 2 \\ 9 & 6 & -2 \end{vmatrix} = 0$ then $x = ?$

- (a) 0 (b) 6 (c) -6 (d) 9

24. The solution set of the equation $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ is

- (a) $\{2, -3, 7\}$ (b) $\{2, 7, -9\}$ (c) $[-2, 3, -7]$ (d) none of these

25. The solution set of the equation $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$ is

- (a) $\{4\}$ (b) $\{2, 4\}$ (c) $\{2, 8\}$ (d) $\{4, 8\}$

26. The solution set of the equation $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$ is

- (a) $\{a, 0\}$ (b) $\{3a, 0\}$ (c) $\{a, 3a\}$ (d) none of these

27. The solution set of the equation $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$ is

- (a) $\left\{\frac{2}{3}, \frac{8}{3}\right\}$ (b) $\left\{\frac{2}{3}, \frac{11}{3}\right\}$ (c) $\left\{\frac{3}{2}, \frac{8}{3}\right\}$ (d) none of these

28. The vertices of a ΔABC are $A(-2, 4)$, $B(2, -6)$ and $C(5, 4)$. The area of ΔABC is

- (a) 17.5 sq units (b) 35 sq units (c) 32 sq units (d) 28 sq units

29. If the points $A(3, -2)$, $B(k, 2)$ and $C(8, 8)$ are collinear then the value of k is

- (a) 2 (b) -3 (c) 5 (d) -4

ANSWERS (OBJECTIVE QUESTIONS)

1. (b) 2. (c) 3. (b) 4. (c) 5. (c) 6. (b) 7. (b) 8. (c) 9. (d)
 10. (b) 11. (c) 12. (a) 13. (b) 14. (b) 15. (c) 16. (b) 17. (c) 18. (c)
 19. (a) 20. (a) 21. (c) 22. (a) 23. (c) 24. (b) 25. (a) 26. (b) 27. (b)
 28. (b) 29. (c)

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

2. $\Delta = (\cos^2 15^\circ - \sin^2 15^\circ) = \cos(2 \times 15^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$.
4. $\Delta = (a + ib)(a - ib) + (c - id)(c + id) = (a^2 + b^2 + c^2 + d^2)$.
5. Apply $R_1 \rightarrow (R_1 + R_2 + R_3)$ and use the result $(1 + \omega + \omega^2) = 0$.
6. $1 + \omega + \omega^2 = 0 \Rightarrow (1 + \omega) = -\omega^2$. Put $(1 + \omega) = -\omega^2$ and expand.
9. $R_1 \rightarrow (R_1 + R_2 + R_3)$ gives all zeros in R_1 and hence $\Delta = 0$.
10. Applying $R_2 \rightarrow (R_2 - 2R_1)$ and $R_3 \rightarrow (R_3 - 3R_1)$ and expanding by C_1 , we get $\Delta = 1$.
12. $C_3 \rightarrow C_3 + (\sin \delta)C_1 - (\cos \delta)C_2$ makes all zeros in C_3 and hence $\Delta = 0$.
13. $C_1 \rightarrow (C_1 + C_2 + C_3)$ and take $(a + b + c)$ common from C_1 .
 Simplify to get

$$\Delta = -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$= -\frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}, \text{ which is negative.}$$
14. Take x common from each of C_2 and C_3 . Apply $R_3 \rightarrow R_3 - 2R_2$.
15. Apply $R_1 \rightarrow (R_1 - R_2)$ and $R_3 \rightarrow (R_3 - R_2)$. Expand by C_3 .
16. Apply $R_2 \rightarrow R_2 - 3R_1$ and $R_3 \rightarrow R_3 - 6R_1$.
17. $\Delta = \begin{vmatrix} b & a & b \\ c & c & a \\ a & b & c \end{vmatrix} + \begin{vmatrix} c & a & b \\ a & c & a \\ b & b & c \end{vmatrix}$
- Apply $R_1 \rightarrow R_1 + R_2 + R_3$ in each and simplify.
19. $\Delta = \begin{vmatrix} bc & b & 1 \\ ca & c & 1 \\ ab & a & 1 \end{vmatrix} + \begin{vmatrix} bc & c & 1 \\ ca & a & 1 \\ ab & b & 1 \end{vmatrix}$.
20. Apply $R_1 \rightarrow R_1 - (R_2 + R_3)$.
 Take (-2) common from R_1 . Apply $R_2 \rightarrow (R_2 - R_1)$ and $R_3 \rightarrow (R_3 - R_1)$.
21. $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$. Take $(a - b)$ common from R_2 and $(c - a)$ common from R_3 .
22. Apply $C_2 \rightarrow (C_2 - C_1)$ and $C_3 \rightarrow (C_3 - C_1)$.
 Now, apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$.
23. Apply $C_1 \rightarrow (C_1 - C_3)$ and take $(x - 7)$ common from C_1 .
25. Apply $C_2 \rightarrow (C_2 - 2C_1)$ and $C_3 \rightarrow (C_3 - 3C_1)$.
 Now, apply $R_2 \rightarrow (R_2 - R_1)$ and $R_3 \rightarrow (R_3 - R_1)$.
26. Apply $C_1 \rightarrow (C_1 + C_2 + C_3)$ and take $(3a - x)$ common from C_1 .

27. Apply $C_1 \rightarrow (C_1 + C_2 + C_3)$ and take $(3x - 2)$ common from C_1 .

$$28. \Delta = \frac{1}{2} \begin{vmatrix} -2 & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} -2 & 4 & 1 \\ 4 & -10 & 0 \\ 7 & 0 & 0 \end{vmatrix} = 35 \text{ sq units.}$$

$$29. \text{ If } \Delta = \begin{vmatrix} 3 & -2 & 1 \\ k & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} \text{ then we must have } \Delta = 0.$$

7. ADJOINT AND INVERSE OF A MATRIX

ADJOINT OF A MATRIX Let $A = [a_{ij}]$ be a square matrix of order n and let A_{ij} denote the cofactor of a_{ij} in $|A|$. Then, the adjoint of A , denoted by $\text{adj } A$, is defined as

$$\text{adj } A = [A_{ji}]_{n \times n}.$$

Thus $\text{adj } A$ is the transpose of the matrix of the corresponding cofactors of elements of $|A|$.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then}$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix},$$

where A_{ij} denotes the cofactor of a_{ij} in $|A|$.

EXAMPLE 1 If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, find $\text{adj } A$.

SOLUTION Clearly, $|A| = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$.

The cofactors of the elements of $|A|$ are given by

$$A_{11} = 3, \quad A_{12} = -1;$$

$$A_{21} = -5, \quad A_{22} = 2$$

$$\therefore \text{adj } A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

EXAMPLE 2 If $A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & 1 \\ -4 & 5 & 3 \end{bmatrix}$, find $\text{adj } A$.

SOLUTION We have, $|A| = \begin{vmatrix} 1 & -2 & 4 \\ 0 & 2 & 1 \\ -4 & 5 & 3 \end{vmatrix}$.

The cofactors of the elements of $|A|$ are given by

$$A_{11} = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 1; \quad A_{12} = -\begin{vmatrix} 0 & 1 \\ -4 & 3 \end{vmatrix} = -4; \quad A_{13} = \begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix} = 8;$$

$$A_{21} = -\begin{vmatrix} -2 & 4 \\ 5 & 3 \end{vmatrix} = 26; \quad A_{22} = \begin{vmatrix} 1 & 4 \\ -4 & 3 \end{vmatrix} = 19; \quad A_{23} = -\begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = 3;$$

$$A_{31} = \begin{vmatrix} -2 & 4 \\ 2 & 1 \end{vmatrix} = -10; \quad A_{32} = -\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = -1; \quad A_{33} = \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = 2.$$

$$\therefore \text{adj } A = \begin{bmatrix} 1 & -4 & 8 \\ 26 & 19 & 3 \\ -10 & -1 & 2 \end{bmatrix}' = \begin{bmatrix} 1 & 26 & -10 \\ -4 & 19 & -1 \\ 8 & 3 & 2 \end{bmatrix}.$$

THEOREM 1 *If A is a square matrix of order n then prove that*

$$A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I.$$

PROOF Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$. Then,

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_{k1} & A_{k2} & \dots & A_{kn} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{k1} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{k2} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{kn} & \dots & A_{nn} \end{bmatrix}.$$

Now, the (i, k) th element of $A \cdot (\text{adj } A) = a_{i1} A_{k1} + a_{i2} A_{k2} + \dots + a_{in} A_{kn}$

$$= \begin{cases} |A|, & \text{when } i = k \\ 0, & \text{when } i \neq k. \end{cases}$$

This shows that each diagonal element of $A \cdot (\text{adj } A)$ is $|A|$ and each one of its nondiagonal elements is 0.

$$\begin{aligned} \therefore A(\text{adj } A) &= \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix} \\ &= |A| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = |A| \cdot I. \end{aligned}$$

Thus, $A(\text{adj } A) = |A| \cdot I$.

Similarly, $(\text{adj } A)A = |A| \cdot I$.

Hence, $A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I$.

SUMMARY

For every square matrix A , we have
 $A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I$.

EXAMPLE 3 If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, verify that

$$A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I.$$

SOLUTION We have

$$A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \Rightarrow |A| = (15 - 7) = 8 \neq 0.$$

The cofactors of the elements of $|A|$ are given by

$$A_{11} = 5, \quad A_{12} = -7;$$

$$A_{21} = -1, \quad A_{22} = 3.$$

$$\therefore (\text{adj } A) = \begin{bmatrix} 5 & -7 \\ -1 & 3 \end{bmatrix}' = \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}.$$

$$\begin{aligned} \therefore A \cdot (\text{adj } A) &= \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 15 - 7 & -3 + 3 \\ 35 - 35 & -7 + 15 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \\ &= 8 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 8I = |A| \cdot I \quad [\because |A| = 8]. \end{aligned}$$

$$\begin{aligned} \text{And, } (\text{adj } A) \cdot A &= \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 15 - 7 & 5 - 5 \\ -21 + 21 & -7 + 15 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \end{aligned}$$

$$= 8 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 8I = |A| \cdot I \quad [\because |A| = 8].$$

Hence, $A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I$.

EXAMPLE 4 If $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$, verify that

$$A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I.$$

SOLUTION We have

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = [1 \cdot (-28 + 30) - 1 \cdot (-18 - 0)] = 20.$$

Now, the cofactors of the elements of $|A|$ are given by

$$A_{11} = \begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix} = 2, A_{12} = -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix} = 21, A_{13} = \begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix} = -18;$$

$$A_{21} = -\begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix} = 6, A_{22} = \begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix} = -7, A_{23} = -\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix} = 6;$$

$$A_{31} = \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} = 4, A_{32} = -\begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = -8, A_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4.$$

$$\therefore \text{adj } A = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}' = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}.$$

$$\begin{aligned} \text{So, } A \cdot (\text{adj } A) &= \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| \cdot I. \end{aligned}$$

Thus, $A \cdot (\text{adj } A) = |A| \cdot I$.

$$\begin{aligned} \text{Further, } (\text{adj } A) \cdot A &= \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| \cdot I. \end{aligned}$$

Thus, $(\text{adj } A) \cdot A = |A| \cdot I$.

Hence, $A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I$.

SINGULAR AND NONSINGULAR MATRICES A square matrix A is said to be

(i) singular if $|A| = 0$, (ii) nonsingular if $|A| \neq 0$.

Examples (i) Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$. Then,

$$|A| = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = (8 - 8) = 0 \text{ and therefore, } A \text{ is singular.}$$

(ii) Let $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then,

$$|B| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (4 - 6) = -2 \neq 0, \text{ and}$$

therefore, B is nonsingular.

INVERTIBLE MATRIX A nonzero square matrix A of order n is said to be invertible if there exists a square matrix B of order n such that $AB = BA = I$.

We say that the inverse of A is B and we write, $A^{-1} = B$.

RESULTS ON INVERTIBLE MATRICES

THEOREM 2 An invertible matrix possesses a unique inverse.

PROOF Let A be an invertible square of order n .

If possible, let B as well as C be the inverse of A .

Then, $AB = BA = I$ and $AC = CA = I$.

Now, $AC = I \Rightarrow B(AC) = B \cdot I = B$.

And, $BA = I \Rightarrow (BA)C = I \cdot C = C$.

But $B(AC) = (BA)C$ [by associative law of multiplication].

$$\therefore B = C.$$

Hence, an invertible has a unique inverse.

THEOREM 3 A square matrix A is invertible if and only if A is nonsingular, i.e., A is invertible $\Leftrightarrow |A| \neq 0$.

PROOF Let A be an invertible square matrix of order n .

Then, there exists a square matrix B of order n such that

$$AB = BA = I.$$

Now, $AB = I \Rightarrow |AB| = |I|$

$$\Rightarrow |A| \cdot |B| = 1 \quad [\because |AB| = |A| \cdot |B| \text{ and } |I| = 1]$$

$$\Rightarrow |A| \neq 0 \quad [\because ab = 1 \Rightarrow a \neq 0 \text{ and } b \neq 0].$$

This shows that A is nonsingular.

Conversely, Let A be nonsingular. Then, $|A| \neq 0$.

$\therefore A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I$ and $|A| \neq 0$

$$\Rightarrow A \cdot \left(\frac{1}{|A|} \cdot \text{adj } A \right) = \left(\frac{1}{|A|} \cdot \text{adj } A \right) \cdot A = I$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \cdot \text{adj } A.$$

This shows that A is invertible.

Hence, A is invertible $\Leftrightarrow A$ is nonsingular.

FORMULA FOR FINDING A^{-1}

Let A be a square matrix such that $|A| \neq 0$.

Then, $A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A).$

EXAMPLE 5 Find the inverse of the matrix, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$.

SOLUTION We have

$$|A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = (14 - 12) = 2 \neq 0.$$

So, A^{-1} exists.

The cofactors of the elements of $|A|$ are given by

$$A_{11} = 7, \quad A_{12} = -(-4) = 4;$$

$$A_{21} = -(-3) = 3, \quad A_{22} = 2.$$

$$\therefore (\text{adj } A) = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}.$$

$$\text{Hence, } A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A)$$

$$= \frac{1}{2} \cdot \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & \frac{3}{2} \\ 2 & 1 \end{bmatrix}.$$

EXAMPLE 6 Find the inverse of the matrix $\begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$.

SOLUTION Let $\begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$. Then,

$$|A| = \begin{vmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 4 & -12 & 2 \\ -4 & 16 & -2 \end{vmatrix}$$

$$\begin{aligned} & [C_1 \rightarrow C_1 + 3C_3 \text{ and } C_2 \rightarrow C_2 - 10C_3] \\ & = (-1) \cdot (64 - 48) = -16 \neq 0. \end{aligned}$$

Thus, $|A| \neq 0$ and therefore, A^{-1} exists.

Now, the cofactors of the elements of $|A|$ are given by

$$A_{11} = \begin{vmatrix} 8 & 2 \\ -4 & -2 \end{vmatrix} = -8, A_{12} = -\begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = 0, A_{13} = \begin{vmatrix} -2 & 8 \\ 2 & -4 \end{vmatrix} = -8;$$

$$A_{21} = -\begin{vmatrix} -10 & -1 \\ -4 & -2 \end{vmatrix} = -16, A_{22} = \begin{vmatrix} 3 & -1 \\ 2 & -2 \end{vmatrix} = -4, A_{23} = -\begin{vmatrix} 3 & -10 \\ 2 & -4 \end{vmatrix} = -8;$$

$$A_{31} = \begin{vmatrix} -10 & -1 \\ 8 & 2 \end{vmatrix} = -12, A_{32} = -\begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} = -4, A_{33} = \begin{vmatrix} 3 & -10 \\ -2 & 8 \end{vmatrix} = 4.$$

$$\therefore (\text{adj } A) = \begin{bmatrix} -8 & 0 & -8 \\ -16 & -4 & -8 \\ -12 & -4 & 4 \end{bmatrix}' = \begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix}.$$

$$\text{Hence, } A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A)$$

$$= \frac{1}{-16} \cdot \begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

Some Results on Invertible Matrices

THEOREM 1 (Cancellation law) Let A, B, C be three square matrices, each of order n such that $AB = AC$. If A is nonsingular then $B = C$.

PROOF Let A, B, C be square matrices, each of order n such that $AB = AC$ and A is nonsingular.

Then, $|A| \neq 0$ and therefore, A^{-1} exists.

$$\begin{aligned} \therefore AB = AC &\Rightarrow A^{-1}(AB) = A^{-1}(AC) \\ &\Rightarrow (A^{-1}A)B = (A^{-1}A)C \\ &\Rightarrow IB = IC \\ &\Rightarrow B = C. \end{aligned}$$

Thus, $AB = AC$ and A is nonsingular $\Rightarrow B = C$.

REMARK If $AB = AC$ and $|A| = 0$ then B and C are not necessarily equal.

Example Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

$$\text{Then, } |A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = (6 - 6) = 0.$$

$$\text{Here, } AB = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$$

$$\text{and } AC = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}.$$

Thus, $AB = AC$ but $B \neq C$.

THEOREM 2 (Reversal law) *If A and B are invertible square matrices of the same order then AB is also invertible and $(AB)^{-1} = B^{-1} A^{-1}$.*

PROOF Let A and B be two invertible square matrices, each of order n .

Then, $|A| \neq 0$ and $|B| \neq 0$.

$\therefore |AB| = |A| \cdot |B| \neq 0$.

Thus, AB is invertible.

$$\begin{aligned} \text{Now, } (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} && [\text{by associativity}] \\ &= (AI)A^{-1} && [\because BB^{-1} = I] \\ &= AA^{-1} = I && [\because AI = A]. \end{aligned}$$

$$\begin{aligned} \text{And, } (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B && [\text{by associativity}] \\ &= B^{-1}(IB) && [\because A^{-1}A = I] \\ &= B^{-1}B = I && [\because IB = B]. \end{aligned}$$

$$\therefore (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I.$$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$.

THEOREM 3 *If A is an invertible square matrix then prove that A' is also invertible and $(A')^{-1} = (A^{-1})'$.*

PROOF Let A be an invertible square matrix of order n .

Then, $|A| \neq 0$.

$\therefore |A'| = |A| \neq 0$.

This shows that A' is invertible.

$$\text{Now, } AA^{-1} = A^{-1}A = I$$

$$\Rightarrow (AA^{-1})' = (A^{-1}A)' = I'$$

$$\Rightarrow (A^{-1})' \cdot A' = A' \cdot (A^{-1})' = I \quad [\because (AB)' = B'A' \text{ and } I' = I]$$

$$\Rightarrow (A')^{-1} = (A^{-1})' \quad [\because AB = BA = I \Rightarrow B^{-1} = A].$$

Hence, $(A')^{-1} = (A^{-1})'$.

THEOREM 4 *If A is an invertible symmetric then prove that A^{-1} is also symmetric.*

PROOF Let A be an invertible symmetric matrix. Then, $A' = A$.

We know that $(A^{-1})' = (A')^{-1}$.

$$\therefore (A^{-1})' = A^{-1} \quad [\because A' = A].$$

Hence, A^{-1} is symmetric.

THEOREM 5 If A and B are nonsingular matrices of the same order then prove that
 $(\text{adj } AB) = (\text{adj } B) \cdot (\text{adj } A).$

PROOF We have

$$|A| \neq 0 \text{ and } |B| \neq 0.$$

$$\therefore |AB| = |A| \cdot |B| \neq 0.$$

So, $(AB)^{-1}$ exists.

$$\text{Now, } A^{-1} = \frac{\text{adj } A}{|A|}; \quad B^{-1} = \frac{\text{adj } B}{|B|} \text{ and } (AB)^{-1} = \frac{\text{adj } (AB)}{|AB|}.$$

$$\therefore \text{adj } (AB) = |AB| \cdot \frac{\text{adj } (AB)}{|AB|} = |A| \cdot |B| \cdot (AB)^{-1}$$

$$[\because |AB| = |A| \cdot |B| \text{ and } \frac{\text{adj } (AB)}{|AB|} = (AB)^{-1}]$$

$$= |A| \cdot |B| \cdot (B^{-1} A^{-1}) \quad [\because (AB)^{-1} = B^{-1} A^{-1}]$$

$$= |A| \cdot |B| \cdot \frac{\text{adj } B}{|B|} \cdot \frac{\text{adj } A}{|A|} = (\text{adj } B) (\text{adj } A).$$

Hence, $\text{adj } (AB) = (\text{adj } B) (\text{adj } A).$

THEOREM 6 For any square matrix A , prove that $(\text{adj } A)' = \text{adj } A'.$

PROOF Let A be a square matrix of order n .

Then, each one of $(\text{adj } A)'$ and $(\text{adj } A')$ is a square matrix of order n .

Also, (i, j) th element of $(\text{adj } A)'$

$$= (j, i) \text{th element of } (\text{adj } A)$$

$$= \text{cofactor of } (i, j) \text{th element of } A$$

$$= \text{cofactor of } (j, i) \text{th element of } A'$$

$$= (i, j) \text{th element of } (\text{adj } A').$$

Hence, $(\text{adj } A)' = (\text{adj } A').$

THEOREM 7 If A is a nonsingular square matrix of order n then prove that
 $|\text{adj } A| = |A|^{n-1}.$

PROOF We have

$$A \cdot (\text{adj } A) = |A| \cdot I$$

$$\Rightarrow |A \cdot (\text{adj } A)| = ||A| \cdot I| \quad [\because A = B \Rightarrow |A| = |B|]$$

$$\Rightarrow |A| |\text{adj } A| = |A|^n |I| \quad [\because |kI| = k^n |I|]$$

$$\Rightarrow |A| \cdot |\text{adj } A| = |A|^n \quad [\because |I| = 1]$$

$$\Rightarrow |\text{adj } A| = |A|^{n-1}$$

Hence, $|\text{adj } A| = |A|^{n-1}.$

THEOREM 8 If A is a nonsingular square matrix of order n then prove that

$$\text{adj}(\text{adj } A) = |A|^{n-2} A.$$

PROOF For any nonsingular B of order n , we have

$$B(\text{adj } B) = |B| \cdot I.$$

Taking $B = \text{adj } A$, we get

$$\begin{aligned} & (\text{adj } A) [\text{adj}(\text{adj } A)] = |\text{adj } A| I \\ \Rightarrow & (\text{adj } A) [\text{adj}(\text{adj } A)] = |A|^{n-1} I \quad [\because |\text{adj } A| = |A|^{n-1}] \\ \Rightarrow & A(\text{adj } A) [\text{adj}(\text{adj } A)] = |A|^{n-1} A \quad [\because A I_n = A] \\ \Rightarrow & |A| I_n [\text{adj}(\text{adj } A)] = |A|^{n-1} A \quad [\because A(\text{adj } A) = |A| I] \\ \Rightarrow & [\text{adj}(\text{adj } A)] = |A|^{n-2} A. \end{aligned}$$

SOLVED EXAMPLES

EXAMPLE 1 If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1} A^{-1}$.

SOLUTION We have $|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = (15 - 14) = 1 \neq 0$.

Cofactors of the elements of $|A|$ are

$$A_{11} = 5, \quad A_{12} = -7;$$

$$A_{21} = -2, \quad A_{22} = 3.$$

$$\therefore \text{adj } A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}' = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}.$$

$$\text{Hence, } A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} \quad [\because |A| = 1].$$

$$\text{Further, } |B| = \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix} = (54 - 56) = -2 \neq 0.$$

Cofactors of the elements of $|B|$ are

$$B_{11} = 9, \quad B_{12} = -8;$$

$$B_{21} = -7, \quad B_{22} = 6.$$

$$\therefore \text{adj } B = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}' = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}.$$

$$\text{Hence, } B^{-1} = \frac{1}{|B|} \cdot \text{adj } B = -\frac{1}{2} \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}.$$

$$\text{Now, } |AB| = |A| \cdot |B| = 1 \times (-2) = -2 \neq 0,$$

$$\text{and } \text{adj } AB = (\text{adj } B) \cdot (\text{adj } A)$$

$$= \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}.$$

$$\therefore (AB)^{-1} = \frac{1}{|AB|} \cdot (\text{adj } AB) = \frac{1}{-2} \cdot \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}.$$

$$\text{Also, } B^{-1}A^{-1} = -\frac{1}{2} \cdot \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = -\frac{1}{2} \cdot \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}.$$

$$\text{Hence, } (AB)^{-1} = B^{-1}A^{-1}.$$

EXAMPLE 2 Show that the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the equation

$$A^2 - 4A - 5I = O, \text{ and hence find } A^{-1}. \quad [\text{CBSE 2008}]$$

SOLUTION We leave it to the reader to show that $A^2 - 4A - 5I = O$.

$$\text{Now, } A^2 - 4A - 5I = O$$

$$\Rightarrow AA - 4A = 5I$$

$$\Rightarrow (AA) \cdot A^{-1} - 4A \cdot A^{-1} = 5I \cdot A^{-1}$$

$$\Rightarrow A(AA^{-1}) - 4I = 5A^{-1}$$

$$\Rightarrow AI - 4I = 5A^{-1}$$

$$\Rightarrow A - 4I = 5A^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I).$$

$$\begin{aligned} \therefore A^{-1} &= \frac{1}{5} \cdot \left\{ \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \\ &= \frac{1}{5} \cdot \left\{ \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right\} \\ &= \frac{1}{5} \cdot \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}. \end{aligned}$$

EXAMPLE 3 Find a matrix X such that $X \cdot \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

SOLUTION Let $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

Here, $|A| = \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = (-3 - 2) = -5 \neq 0$.

So, A is nonsingular and therefore, invertible.

The given equation is $XA = B$.

$$\begin{aligned} \text{Now, } XA = B &\Rightarrow (XA) \cdot A^{-1} = BA^{-1} \\ &\Rightarrow X(AA^{-1}) = BA^{-1} \\ &\Rightarrow XI = BA^{-1} \\ &\Rightarrow X = BA^{-1}. \end{aligned}$$

Now, cofactors of elements of $|A|$ are

$$A_{11} = -1, \quad A_{12} = -1;$$

$$A_{21} = -2, \quad A_{22} = 3.$$

$$\therefore \text{adj } A = \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix}' = \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}.$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = -\frac{1}{5} \cdot \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}.$$

Hence, $X = BA^{-1}$

$$\begin{aligned} &= \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \cdot \left(-\frac{1}{5}\right) \cdot \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix} \\ &= \left(-\frac{1}{5}\right) \cdot \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix} \\ &= \left(-\frac{1}{5}\right) \cdot \begin{bmatrix} -5 & -5 \\ -5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \end{aligned}$$

EXAMPLE 4 Find the matrix A satisfying the equation

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot A \cdot \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

SOLUTION Let $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$.

Clearly, $|B| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = (4 - 3) = 1 \neq 0$.

And, $|C| = \begin{vmatrix} -3 & 2 \\ 5 & -3 \end{vmatrix} = (9 - 10) = -1 \neq 0$.

This shows that B as well as C is invertible.

The given matrix equation is $BAC = I$.

$$\begin{aligned} \text{Now, } BAC = I &\Rightarrow B^{-1} BAC C^{-1} = B^{-1} I C^{-1} \\ &\Rightarrow I A I = B^{-1} C^{-1} \\ &\Rightarrow A = B^{-1} C^{-1}. \end{aligned}$$

Now, the cofactors of the elements of $|B|$ are

$$B_{11} = 2, B_{12} = -3; B_{21} = -1, B_{22} = 2.$$

$$\therefore \text{adj } B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

$$\text{So, } B^{-1} = \frac{1}{|B|} \cdot \text{adj } B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \quad [\because |B| = 1].$$

Again, the cofactors of the elements of $|C|$ are

$$C_{11} = -3, C_{12} = -5; C_{21} = -2, C_{22} = -3.$$

$$\therefore \text{adj } C = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}' = \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$$\Rightarrow C^{-1} = \frac{1}{|C|} \cdot (\text{adj } C) = \frac{1}{(-1)} \cdot \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\Rightarrow A = (B^{-1} C^{-1}) = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

EXAMPLE 5 If $A = \begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$, verify that $(\text{adj } A)^{-1} = (\text{adj } A^{-1})$.

SOLUTION We have, $|A| = \begin{vmatrix} 2 & -3 \\ 4 & 6 \end{vmatrix} = (12 + 12) = 24 \neq 0$.

Cofactors of the elements of $|A|$ are

$$A_{11} = 6, A_{12} = -4; A_{21} = 3, A_{22} = 2.$$

$$\therefore \text{adj } A = \begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}.$$

$$\text{So, } A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{24} \cdot \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ -\frac{1}{6} & \frac{1}{12} \end{bmatrix}.$$

$$\text{Now, } |A^{-1}| = \begin{vmatrix} \frac{1}{4} & \frac{1}{8} \\ -\frac{1}{6} & \frac{1}{12} \end{vmatrix} = \left(\frac{1}{48} + \frac{1}{48} \right) = \frac{1}{24}.$$

Cofactors of the elements of $|A^{-1}|$ are

$$C_{11} = \frac{1}{12}, C_{12} = \frac{1}{6}; C_{21} = -\frac{1}{8}, C_{22} = \frac{1}{4}.$$

$$\therefore \operatorname{adj} A^{-1} = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ -\frac{1}{8} & \frac{1}{4} \end{bmatrix}' = \begin{bmatrix} \frac{1}{12} & \frac{-1}{8} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix}.$$

$$\therefore (\operatorname{adj} A)(\operatorname{adj} A^{-1}) = \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{12} & \frac{-1}{8} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$\text{And, } (\operatorname{adj} A^{-1})(\operatorname{adj} A) = \begin{bmatrix} \frac{1}{12} & \frac{-1}{8} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Thus, $(\operatorname{adj} A)(\operatorname{adj} A^{-1}) = (\operatorname{adj} A^{-1})(\operatorname{adj} A) = I$.

Hence, $(\operatorname{adj} A)^{-1} = (\operatorname{adj} A^{-1})$.

EXAMPLE 6 If $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that $(\operatorname{adj} A)^{-1} = (\operatorname{adj} A^{-1})$.

SOLUTION We have

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & -1 & 3 \\ 0 & 3 & 4 \end{vmatrix} \begin{bmatrix} R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{bmatrix} \\ &= 1 \cdot (-4 - 9) = -13 \neq 0. \end{aligned}$$

So, A^{-1} exists.

The cofactors of the elements of $|A|$ are

$$A_{11} = \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 14, A_{12} = -\begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} = 11, A_{13} = \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = -5;$$

$$A_{21} = -\begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} = 11, A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4, A_{23} = -\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = -3;$$

$$A_{31} = \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = -5, A_{32} = -\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -3, A_{33} = \begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix} = -1.$$

$$\therefore (\operatorname{adj} A) = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}' = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A) = \frac{1}{-13} \cdot \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{-14}{13} & \frac{-11}{13} & \frac{5}{13} \\ \frac{-11}{13} & \frac{-4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}.$$

The cofactors of the elements of $|A^{-1}|$ are

$$C_{11} = \begin{vmatrix} \frac{-4}{13} & \frac{3}{13} \\ \frac{3}{13} & \frac{1}{13} \end{vmatrix} = \frac{-1}{13}, C_{12} = - \begin{vmatrix} \frac{-11}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{1}{13} \end{vmatrix} = \frac{2}{13},$$

$$C_{13} = \begin{vmatrix} \frac{-11}{13} & \frac{-4}{13} \\ \frac{5}{13} & \frac{3}{13} \end{vmatrix} = \frac{-1}{13};$$

$$C_{21} = - \begin{vmatrix} \frac{-11}{13} & \frac{5}{13} \\ \frac{3}{13} & \frac{1}{13} \end{vmatrix} = \frac{2}{13}, C_{22} = \begin{vmatrix} \frac{-14}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{1}{13} \end{vmatrix} = \frac{-3}{13},$$

$$C_{23} = - \begin{vmatrix} \frac{-14}{13} & \frac{-11}{13} \\ \frac{5}{13} & \frac{3}{13} \end{vmatrix} = \frac{-1}{13};$$

$$C_{31} = \begin{vmatrix} \frac{-11}{13} & \frac{5}{13} \\ \frac{-4}{13} & \frac{3}{13} \end{vmatrix} = \frac{-1}{13}, C_{32} = - \begin{vmatrix} \frac{-14}{13} & \frac{5}{13} \\ \frac{-11}{13} & \frac{3}{13} \end{vmatrix} = \frac{-1}{13},$$

$$C_{33} = \begin{vmatrix} \frac{-14}{13} & \frac{-11}{13} \\ \frac{-11}{13} & \frac{-4}{13} \end{vmatrix} = \frac{-5}{13}.$$

$$\therefore (\text{adj } A^{-1}) = \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & \frac{-1}{13} \\ \frac{2}{13} & \frac{-3}{13} & \frac{-1}{13} \\ \frac{-1}{13} & \frac{-1}{13} & \frac{-5}{13} \end{bmatrix}' = \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & \frac{-1}{13} \\ \frac{2}{13} & \frac{-3}{13} & \frac{-1}{13} \\ \frac{-1}{13} & \frac{-1}{13} & \frac{-5}{13} \end{bmatrix}.$$

$$\begin{aligned} \therefore (\text{adj } A)(\text{adj } A^{-1}) &= \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & \frac{-1}{13} \\ \frac{2}{13} & \frac{-3}{13} & \frac{-1}{13} \\ \frac{-1}{13} & \frac{-1}{13} & \frac{-5}{13} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus, $(\text{adj } A)(\text{adj } A^{-1}) = I$. Similarly, $(\text{adj } A^{-1})(\text{adj } A) = I$.

Hence, $(\text{adj } A)^{-1} = (\text{adj } A^{-1})$.

EXAMPLE 7 If $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, verify that $(A')^{-1} = (A^{-1})'$.

SOLUTION Given: $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, and therefore $A' = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$.

$$\therefore |A| = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = (10 - 3) = 7 \neq 0.$$

So, A^{-1} exists.

The cofactors of the elements of $|A|$ are

$$A_{11} = 5, A_{12} = -1; A_{21} = -3, A_{22} = 2$$

$$\therefore (\text{adj } A) = \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A) = \frac{1}{7} \cdot \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} & \frac{-3}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$$

$$\Rightarrow (A^{-1})' = \begin{bmatrix} \frac{5}{7} & \frac{-1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{bmatrix}. \quad \dots (i)$$

Also, $|A'| = |A| = 7 \neq 0$.

So, $(A')^{-1}$ exists.

The cofactors of elements of $|A'|$ are

$$C_{11} = 5, C_{12} = -3; C_{21} = -1, C_{22} = 2.$$

$$\therefore (\text{adj } A') = \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}' = \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow (A')^{-1} = \frac{1}{|A'|} \cdot (\text{adj } A') = \frac{1}{7} \cdot \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow (A')^{-1} = \begin{bmatrix} \frac{5}{7} & \frac{-1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{bmatrix}. \quad \dots \text{(ii)}$$

Hence, from (i) and (ii), we get $(A')^{-1} = (A^{-1})'$.

EXAMPLE 8 If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

SOLUTION We have

$$|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = (1 + \tan^2 x) = \sec^2 x \neq 0.$$

So, A is invertible.

The cofactors of the elements of $|A|$ are

$$A_{11} = 1, A_{12} = \tan x; A_{21} = -\tan x, A_{22} = 1.$$

$$\therefore (\text{adj } A) = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}' = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow A^{-1} &= \frac{1}{|A|} (\text{adj } A) = \frac{1}{\sec^2 x} \cdot \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \\ &= \cos^2 x \cdot \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} = \begin{bmatrix} \cos^2 x & -\tan x \cos^2 x \\ \tan x \cos^2 x & \cos^2 x \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 x & -\sin x \cos x \\ \sin x \cos x & \cos^2 x \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \Rightarrow A'A^{-1} &= \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \begin{bmatrix} \cos^2 x & -\sin x \cos x \\ \sin x \cos x & \cos^2 x \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 x - \sin^2 x & -2\sin x \cos x \\ 2\sin x \cos x & -\sin^2 x + \cos^2 x \end{bmatrix} \\ &= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}. \end{aligned}$$

$$\text{Hence, } A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}.$$

EXAMPLE 9 Let $F(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $G(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$.

Show that $[F(\alpha) \cdot G(\beta)]^{-1} = G(-\beta) \cdot F(-\alpha)$.

SOLUTION We have

$$\begin{aligned} F(\alpha) \cdot F(-\alpha) &= \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \alpha + \sin^2 \alpha & 0 & 0 \\ 0 & \sin^2 \alpha + \cos^2 \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I. \end{aligned}$$

Thus, $F(\alpha) \cdot F(-\alpha) = I \Rightarrow \{F(\alpha)\}^{-1} = F(-\alpha)$.

Similarly, $G(\beta) \cdot G(-\beta) = I \Rightarrow \{G(\beta)\}^{-1} = G(-\beta)$.

$$\begin{aligned} \therefore \{F(\alpha) \cdot G(\beta)\}^{-1} &= \{G(\beta)\}^{-1} \cdot \{F(\alpha)\}^{-1} \quad [\text{by reversal law}] \\ &= G(-\beta) \cdot F(-\alpha). \end{aligned}$$

Hence, $\{F(\alpha) \cdot G(\beta)\}^{-1} = G(-\beta) \cdot F(-\alpha)$.

EXERCISE 7

Find the adjoint of the given matrix and verify in each case that $A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I$.

1. $\begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 5 & 3 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{bmatrix}$

9. $\begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, show that $\text{adj } A = A$.

11. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, show that $\text{adj } A = 3A'$.

Find the inverse of each of the matrices given below:

12. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$

15. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, when $(ad - bc) \neq 0$

16. $\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

17. $\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 2 & 6 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$

20. $\begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

21. $\begin{bmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{bmatrix}$

22. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19} A$.

23. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, show that $A^{-1} = A^2$.

24. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, prove that $A^{-1} = A^3$.

25. If $A = \frac{1}{9} \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, show that $A^{-1} = A'$.

26. Let $D = \text{diag } [d_1, d_2, d_3]$, where none of d_1, d_2, d_3 is 0; prove that $D^{-1} = \text{diag } [d_1^{-1}, d_2^{-1}, d_3^{-1}]$.

27. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1} A^{-1}$.

28. If $A = \begin{bmatrix} 9 & -1 \\ 6 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 3 \\ 5 & -4 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1} A^{-1}$.

29. Compute $(AB)^{-1}$ when $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$.

30. Obtain the inverses of the matrices $\begin{bmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}$.

And, hence find the inverse of the matrix $\begin{bmatrix} 1+pq & p & 0 \\ q & 1+pq & p \\ 0 & q & 1 \end{bmatrix}$.

31. If $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$, verify that $A^2 - 4A - I = O$, and hence find A^{-1} .

32. Show that the matrix $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ satisfies the equation $x^2 + 4x - 42 = 0$ and hence find A^{-1} .

33. If $A = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$, show that $A^2 + 3A + 4I_2 = O$ and hence find A^{-1} .

34. If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find x and y such that $A^2 + xI = yA$. Hence, find A^{-1} .

[CBSE 2005]

35. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find the value of λ so that $A^2 = \lambda A - 2I$. Hence, find A^{-1} .

[CBSE 2007]

36. Show that the $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ satisfies the equation

$A^3 - A^2 - 3A - I = O$, and hence find A^{-1} .

37. Prove that: (i) $\text{adj } I = I$ (ii) $\text{adj } O = O$ (iii) $I^{-1} = I$.

ANSWERS (EXERCISE 7)

1. $\begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$

4. $\begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

6. $\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

7. $\begin{bmatrix} -34 & 10 & 31 \\ 14 & 0 & -21 \\ 46 & -30 & -44 \end{bmatrix}$

8. $\begin{bmatrix} -42 & -24 & 30 \\ 3 & 30 & -21 \\ 7 & -18 & -5 \end{bmatrix}$

9. $\begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} \frac{3}{10} & \frac{-1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$

14. $\begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ -\frac{1}{6} & \frac{1}{12} \end{bmatrix}$

15. $\frac{1}{(ad-bc)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

16. $\frac{1}{27} \cdot \begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$

17. $\frac{1}{32} \cdot \begin{bmatrix} 6 & 6 & 1 \\ -2 & -2 & 5 \\ 8 & -14 & 3 \end{bmatrix}$

18. $\frac{1}{5} \cdot \begin{bmatrix} -2 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$

19. $\frac{1}{4} \cdot \begin{bmatrix} -8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$

20. $\frac{1}{19} \cdot \begin{bmatrix} 1 & -6 & 1 \\ -5 & -8 & 14 \\ 3 & 1 & 3 \end{bmatrix}$

21. $\frac{1}{10} \cdot \begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}$

29. $\frac{1}{19} \cdot \begin{bmatrix} 16 & 12 & 1 \\ 21 & 11 & -7 \\ 10 & -2 & 3 \end{bmatrix}$

30. $\begin{bmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ q^2 & -q & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -p & p^2 \\ -q & pq+1 & -qp^2-p \\ q^2 & -pq^2-q & p^2q^2+pq+1 \end{bmatrix}$

31. $\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$

32. $\frac{1}{42} \cdot \begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$

33. $A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} \end{bmatrix}$

34. $x = 8, y = 8 \text{ and } A^{-1} = \frac{1}{8} \cdot \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$

35. $\lambda = 1, A^{-1} = \frac{1}{2} \cdot \begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$

36. $A^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 7)

29. Find A^{-1} . Then, $(AB)^{-1} = B^{-1}A^{-1}$.

30. Let the first two matrices be A and B . Then, the third matrix is AB . Now,
 $(AB)^{-1} = (B^{-1} \cdot A^{-1})$.

37. (i) $|I| = 1$. Cofactors of a_{ij} in $|I|$ are $A_{ij} = \begin{cases} 1, & \text{when } i = j \\ 0, & \text{when } i \neq j \end{cases}$

$$\therefore (\text{adj } I) = [A_{ij}]' = I' = I.$$

(ii) $|O| = 0$ and cofactor of each element of $|O|$ is 0.

$$\therefore \text{adj } O = O.$$

(iii) $I \cdot I = I \Rightarrow I^{-1} = I.$

8. SYSTEM OF LINEAR EQUATIONS

Solving a System of Linear Equations by Matrix Method

CONSISTENT SYSTEM OF EQUATIONS A given system of equations is said to be consistent if it has one or more solutions.

INCONSISTENT SYSTEM OF EQUATIONS A given system of equations is said to be inconsistent if it has no solution.

Consider the system of equations

$$\begin{aligned}a_1x + b_1y + c_1z &= d_1, \\a_2x + b_2y + c_2z &= d_2, \\a_3x + b_3y + c_3z &= d_3.\end{aligned}$$

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Then, the given system can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

$$\therefore AX = B.$$

CASE 1 When $|A| \neq 0$

In this case, A^{-1} exists.

$$\therefore AX = B \Rightarrow A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow (A^{-1}A)X = A^{-1}B \quad [\text{by associative law}]$$

$$\Rightarrow I \cdot X = A^{-1}B \quad [\because A^{-1}A = I]$$

$$\Rightarrow X = A^{-1}B.$$

Since A^{-1} is unique, the given system has a unique solution.

Thus, when $|A| \neq 0$, then the given system is consistent and it has a unique solution.

CASE 2 When $|A| = 0$ and $(\text{adj } A)B \neq O$

In this case, the given system has no solution and hence it is inconsistent.

CASE 3 When $|A| = 0$ and $(\text{adj } A)B = O$

In this case, the given system has infinitely many solutions.

SUMMARY

Let $AX = B$ be the given system of equations.

- (i) If $|A| \neq 0$, the system has a unique solution.
- (ii) If $|A| = 0$ and $(\text{adj } A)B \neq O$ then the given system has no solution.
- (iii) If $|A| = 0$ and $(\text{adj } A)B = O$ then the system has infinitely many solutions.

SOLVED EXAMPLES

EXAMPLE 1 Use matrix method to show that the system of equations

$$\begin{aligned} 2x + 5y &= 7, \\ 6x + 15y &= 13 \end{aligned}$$

is inconsistent.

SOLUTION The given equations are

$$2x + 5y = 7, \quad \dots (i)$$

$$6x + 15y = 13. \quad \dots (ii)$$

Let $A = \begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$.

Then, the given system in matrix form is $AX = B$.

Now, $|A| = \begin{vmatrix} 2 & 5 \\ 6 & 15 \end{vmatrix} = 0$.

The system will be inconsistent if $(\text{adj } A)B \neq O$.

The minors of the elements of $|A|$ are

$$M_{11} = 15, \quad M_{12} = 6;$$

$$M_{21} = 5, \quad M_{22} = 2.$$

So, the cofactors of the elements of $|A|$ are

$$A_{11} = 15, \quad A_{12} = -6;$$

$$A_{21} = -5, \quad A_{22} = 2.$$

$$\therefore \text{adj } A = \begin{bmatrix} 15 & -6 \\ -5 & 2 \end{bmatrix}' = \begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix}$$

$$\Rightarrow (\text{adj } A)B = \begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 13 \end{bmatrix} = \begin{bmatrix} 105 - 65 \\ -42 + 26 \end{bmatrix} = \begin{bmatrix} 40 \\ -16 \end{bmatrix} \neq O.$$

Thus, $|A| = 0$ and $(\text{adj } A)B \neq O$.

Hence, the given system of equations is inconsistent.

EXAMPLE 2 Use matrix method to show that the following system of equations is inconsistent:

$$\begin{aligned} 3x - y + 2z &= 3, \\ 2x + y + 3z &= 5, \\ x - 2y - z &= 1. \end{aligned}$$

SOLUTION Let us take

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}.$$

The given system in matrix form is $AX = B$.

$$\begin{aligned} \text{Now, } |A| &= \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & -1 \end{vmatrix} \\ &= 3(-1 + 6) + 1(-2 - 3) + 2(-4 - 1) \\ &= (15 - 5 - 10) = 0. \end{aligned}$$

So, the system will be inconsistent if $(\text{adj } A) \cdot B \neq O$.

The minors of the elements of $|A|$ are

$$\begin{aligned} M_{11} &= 5, \quad M_{12} = -5, \quad M_{13} = -5; \\ M_{21} &= 5, \quad M_{22} = -5, \quad M_{23} = -5; \\ M_{31} &= -5, \quad M_{32} = 5, \quad M_{33} = 5. \end{aligned}$$

So, the cofactors of the elements of $|A|$ are

$$\begin{aligned} A_{11} &= 5, \quad A_{12} = 5, \quad A_{13} = -5; \\ A_{21} &= -5, \quad A_{22} = -5, \quad A_{23} = 5; \\ A_{31} &= -5, \quad A_{32} = -5, \quad A_{33} = 5. \end{aligned}$$

$$\therefore (\text{adj } A) = \begin{bmatrix} 5 & 5 & -5 \\ -5 & -5 & 5 \\ -5 & -5 & 5 \end{bmatrix}' = \begin{bmatrix} 5 & -5 & -5 \\ 5 & -5 & -5 \\ -5 & 5 & 5 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow (\text{adj } A)B &= \begin{bmatrix} 5 & -5 & -5 \\ 5 & -5 & -5 \\ -5 & 5 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 \cdot 3 + (-5) \cdot 5 + (-5) \cdot 1 \\ 5 \cdot 3 + (-5) \cdot 5 + (-5) \cdot 1 \\ (-5) \cdot 3 + 5 \cdot 5 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 15 - 25 - 5 \\ 15 - 25 - 5 \\ -15 + 25 + 5 \end{bmatrix} \\ &= \begin{bmatrix} -15 \\ -15 \\ 15 \end{bmatrix} \neq O. \end{aligned}$$

Thus, $|A| = 0$ and $(\text{adj } A)B \neq O$.

Hence, the given system of equations is inconsistent.

EXAMPLE 3 Show that the following system of equations is consistent and solve it:

$$\begin{aligned} 2x + 5y &= 1, \\ 3x + 2y &= 7. \end{aligned}$$

SOLUTION The given system of equations is

$$2x + 5y = 1, \quad \dots (i)$$

$$3x + 2y = 7. \quad \dots (ii)$$

$$\text{Let } A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

Then, the given system is $AX = B$.

$$\text{Now, } |A| = \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} = (4 - 15) = -11 \neq 0.$$

Hence, the given system has a unique solution.

The minors of the elements of $|A|$ are

$$M_{11} = 2, \quad M_{12} = 3;$$

$$M_{21} = 5, \quad M_{22} = 2.$$

So, the cofactors of the elements of $|A|$ are

$$A_{11} = 2, \quad A_{12} = -3;$$

$$A_{21} = -5, \quad A_{22} = 2.$$

$$\therefore (\text{adj } A) = \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} (\text{adj } A) = \frac{-1}{11} \cdot \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{-2}{11} & \frac{5}{11} \\ \frac{3}{11} & \frac{-2}{11} \end{bmatrix}$$

$$\Rightarrow X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-2}{11} & \frac{5}{11} \\ \frac{3}{11} & \frac{-2}{11} \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{-2}{11} + \frac{35}{11} \\ \frac{3}{11} - \frac{14}{11} \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\Rightarrow x = 3 \text{ and } y = -1.$$

Hence, $x = 3$ and $y = -1$.

EXAMPLE 4 If $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$, find A^{-1} and hence solve the system of linear

equations:

$$x + 2y - 3z = -4; \quad 2x + 3y + 2z = 2; \quad 3x - 3y - 4z = 11. \quad [\text{CBSE 2012C}]$$

SOLUTION The given equations are $x + 2y - 3z = -4. \quad \dots (i)$

$$2x + 3y + 2z = 2, \quad \dots (ii)$$

$$3x - 3y - 4z = 11. \quad \dots (iii)$$

$$\text{Let } A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}.$$

So, the given system in matrix form is $AX = B$.

$$\begin{aligned} \text{Now, } |A| &= \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -1 & 8 \\ 0 & -9 & 5 \end{vmatrix} \\ &= 1 \cdot (-5 + 72) = 67 \neq 0. \end{aligned}$$

Thus, A is invertible.

So, the system has a unique solution, $X = A^{-1}B$.

Now, the cofactors of the elements of $|A|$ are

$$\begin{aligned} A_{11} &= -6, & A_{12} &= 14, & A_{13} &= -15; \\ A_{21} &= 17, & A_{22} &= 5, & A_{23} &= 9; \\ A_{31} &= 13, & A_{32} &= -8, & A_{33} &= -1. \end{aligned}$$

$$\therefore \text{adj } A = \begin{bmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{bmatrix}' = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}.$$

$$\text{So, } A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{67} \cdot \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}.$$

$$\therefore X = A^{-1} B$$

$$\begin{aligned} \text{or } \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{67} \cdot \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix} \\ &= \frac{1}{67} \cdot \begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}. \end{aligned}$$

$$\therefore x = 3, \quad y = -2 \quad \text{and} \quad z = 1.$$

EXAMPLE 5 Using matrices, solve the following system of linear equations:

$$\begin{aligned} 3x + 4y + 2z &= 8, \\ 2y - 3z &= 3, \\ x - 2y + 6z &= -2. \end{aligned}$$

[CBSE 2006C]

SOLUTION The given equations are

$$3x + 4y + 2z = 8, \quad \dots \text{ (i)}$$

$$2y - 3z = 3, \quad \dots \text{ (ii)}$$

$$x - 2y + 6z = -2. \quad \dots \text{ (iii)}$$

$$\text{Let } A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}.$$

So, the given system in matrix form is $AX = B$.

$$\begin{aligned} \text{Now, } |A| &= \begin{vmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 10 & -16 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{vmatrix} \quad [R_1 \rightarrow R_1 - 3R_3] \\ &= 1 \cdot (-30 + 32) = 2 \neq 0. \end{aligned}$$

So, A is invertible.

Therefore, the given system has a unique solution, $X = A^{-1}B$.

Now, the minors of the elements of $|A|$ are

$$M_{11} = 6, \quad M_{12} = 3, \quad M_{13} = -2;$$

$$M_{21} = 28, \quad M_{22} = 16, \quad M_{23} = -10;$$

$$M_{31} = -16, \quad M_{32} = -9, \quad M_{33} = 6.$$

The cofactors of the elements of $|A|$ are

$$A_{11} = 6, \quad A_{12} = -3, \quad A_{13} = -2;$$

$$A_{21} = -28, \quad A_{22} = 16, \quad A_{23} = 10;$$

$$A_{31} = -16, \quad A_{32} = 9, \quad A_{33} = 6.$$

$$\therefore (\text{adj } A) = \begin{bmatrix} 6 & -3 & -2 \\ -28 & 16 & 10 \\ -16 & 9 & 6 \end{bmatrix}' = \begin{bmatrix} 6 & -28 & -16 \\ -3 & 16 & 9 \\ -2 & 10 & 6 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A)$$

$$= \frac{1}{2} \cdot \begin{bmatrix} 6 & -28 & -16 \\ -3 & 16 & 9 \\ -2 & 10 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -14 & -8 \\ -\frac{3}{2} & 8 & \frac{9}{2} \\ -1 & 5 & 3 \end{bmatrix}.$$

$$\therefore X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -14 & -8 \\ -\frac{3}{2} & 8 & \frac{9}{2} \\ -1 & 5 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 - 42 + 16 \\ -12 + 24 - 9 \\ -8 + 15 - 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow x = -2, \quad y = 3 \quad \text{and} \quad z = 1.$$

Hence, $x = -2$, $y = 3$ and $z = 1$.

EXAMPLE 6 Using matrices, solve the following system of equations:

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4;$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1;$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2 \quad (x, y, z \neq 0)$$

[CBSE 2011]

SOLUTION Putting $\frac{1}{x} = u$, $\frac{1}{y} = v$ and $\frac{1}{z} = w$, the given equations become:

$$2u + 3v + 10w = 4, \quad \dots \text{(i)}$$

$$4u - 6v + 5w = 1, \quad \dots \text{(ii)}$$

$$6u + 9v - 20w = 2. \quad \dots \text{(iii)}$$

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, \quad Y = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Then, the given system in matrix form is $AY = B$.

$$\begin{aligned} \text{Now, } |A| &= \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 3 & 10 \\ 0 & -12 & -15 \\ 0 & 0 & -50 \end{vmatrix} \quad [R_2 \rightarrow R_2 - 2R_1; R_3 \rightarrow R_3 - 3R_1] \\ &= (-50) \cdot (-24 - 0) = 1200 \neq 0. \end{aligned}$$

Thus, A is invertible.

So, the given system has a unique solution, $Y = A^{-1}B$.

The minors of the elements of $|A|$ are

$$M_{11} = 75, \quad M_{12} = -110, \quad M_{13} = 72;$$

$$M_{21} = -150, \quad M_{22} = -100, \quad M_{23} = 0;$$

$$M_{31} = 75, \quad M_{32} = -30, \quad M_{33} = -24.$$

So, the cofactors of the elements of $|A|$ are

$$A_{11} = 75, \quad A_{12} = 110, \quad A_{13} = 72;$$

$$A_{21} = 150, \quad A_{22} = -100, \quad A_{23} = 0;$$

$$A_{31} = 75, \quad A_{32} = 30, \quad A_{33} = -24.$$

$$\therefore (\text{adj } A) = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}' = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A) = \frac{1}{1200} \cdot \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

$$\therefore Y = A^{-1}B$$

$$\begin{aligned} \Rightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix} &= \frac{1}{1200} \cdot \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{1200} \cdot \begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix} = \frac{1}{1200} \cdot \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} \\ &= \begin{bmatrix} \frac{600}{1200} \\ \frac{400}{1200} \\ \frac{240}{1200} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix} \end{aligned}$$

$$\Rightarrow u = \frac{1}{2}, v = \frac{1}{3}, w = \frac{1}{5}$$

$$\Rightarrow \frac{1}{x} = \frac{1}{2}, \frac{1}{y} = \frac{1}{3} \text{ and } \frac{1}{z} = \frac{1}{5}$$

$$\Rightarrow x = 2, y = 3 \text{ and } z = 5.$$

$$\text{Hence } x = 2, y = 3 \text{ and } z = 5.$$

EXAMPLE 7 Use the product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the

following system of equations:

$$x - y + 2z = 1;$$

$$2y - 3z = 1;$$

$$3x - 2y + 4z = 2.$$

SOLUTION The given equations are

$$x - y + 2z = 1, \quad \dots \text{(i)}$$

$$2y - 3z = 1, \quad \dots \text{(ii)}$$

$$3x - 2y + 4z = 2. \quad \dots \text{(iii)}$$

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Then, the given system in matrix form is $AX = B$.

$$\begin{aligned}
 \text{Now, } & \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} \\
 &= \begin{bmatrix} -2 - 9 + 12 & 0 - 2 + 2 & 1 + 3 - 4 \\ 0 + 18 - 18 & 0 + 4 - 3 & 0 - 6 + 6 \\ -6 - 18 + 24 & 0 - 4 + 4 & 3 + 6 - 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 \Rightarrow & A \cdot \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} = I \\
 \Rightarrow & A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}.
 \end{aligned}$$

Now, $AX = B$

$$\Rightarrow X = A^{-1}B$$

$$\begin{aligned}
 \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} -2 + 0 + 2 \\ 9 + 2 - 6 \\ 6 + 1 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}
 \end{aligned}$$

$$\Rightarrow x = 0, y = 5 \text{ and } z = 3.$$

Hence, $x = 0, y = 5$ and $z = 3$.

EXAMPLE 8 Given $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$, find AB and

use this result in solving the following system of equations:

$$\begin{aligned}
 x - y + z &= 4; \\
 x - 2y - 2z &= 9; \\
 2x + y + 3z &= 1.
 \end{aligned}$$

[CBSE 2006C, '10C, '12C]

SOLUTION The given equations are

$$x - y + z = 4, \quad \dots \text{(i)}$$

$$x - 2y - 2z = 9, \quad \dots \text{(ii)}$$

$$2x + y + 3z = 1. \quad \dots \text{(iii)}$$

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } C = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}.$$

Then, the given system of equations is $AX = C$.

$$\begin{aligned}
 \text{Now, } AB &= \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \\
 &= 8I \\
 \Rightarrow A \cdot \left(\frac{1}{8} B \right) &= I
 \end{aligned}$$

$$\Rightarrow A^{-1} = \frac{1}{8} B = \frac{1}{8} \cdot \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}.$$

Now, $AX = C$

$$\Rightarrow X = A^{-1}C$$

$$\begin{aligned}
 \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{8} \cdot \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} \\
 &= \frac{1}{8} \cdot \begin{bmatrix} -16 + 36 + 4 \\ -28 + 9 + 3 \\ 20 - 27 - 1 \end{bmatrix} = \frac{1}{8} \cdot \begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}
 \end{aligned}$$

$$\Rightarrow x = 3, y = -2 \text{ and } z = -1.$$

Hence, $x = 3, y = -2$ and $z = -1$.

EXAMPLE 9 The sum of three numbers is 6. Twice the third number when added to the first number gives 7. On adding the sum of the second and third numbers to thrice the first number, we get 12. Find the numbers, using method.

SOLUTION Let the first, second and third numbers be x, y, z respectively. Then,

$$x + y + z = 6, \quad \dots \text{(i)}$$

$$x + 2z = 7, \quad \dots \text{(ii)}$$

$$3x + y + z = 12. \quad \dots \text{(iii)}$$

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}.$$

Then, the given system in matrix form is $AX = B$.

$$\begin{aligned} \text{Now, } |A| &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & -2 \end{vmatrix} \quad \begin{bmatrix} R_2 \rightarrow R_2 - R_1; \\ R_3 \rightarrow R_3 - 3R_1 \end{bmatrix} \\ &= 1 \cdot (2 + 2) = 4 \neq 0. \end{aligned}$$

$\therefore A$ is invertible.

So, the given system has a unique solution, $X = A^{-1}B$.

The minors of the elements of $|A|$ are

$$M_{11} = -2, \quad M_{12} = -5, \quad M_{13} = 1;$$

$$M_{21} = 0, \quad M_{22} = -2, \quad M_{23} = -2;$$

$$M_{31} = 2, \quad M_{32} = 1, \quad M_{33} = -1.$$

The cofactors of the elements of $|A|$ are

$$A_{11} = -2, \quad A_{12} = 5, \quad A_{13} = 1;$$

$$A_{21} = 0, \quad A_{22} = -2, \quad A_{23} = 2;$$

$$A_{31} = 2, \quad A_{32} = -1, \quad A_{33} = -1.$$

$$\therefore (\text{adj } A) = \begin{bmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{bmatrix}' = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A) = \frac{1}{4} \cdot \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\Rightarrow X = A^{-1}B$$

$$\begin{aligned} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{4} \cdot \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix} \\ &= \frac{1}{4} \cdot \begin{bmatrix} -12 + 0 + 24 \\ 30 - 14 - 12 \\ 6 + 14 - 12 \end{bmatrix} = \frac{1}{4} \cdot \begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

$$\Rightarrow x = 3, \quad y = 1, \quad z = 2.$$

Hence, the required numbers are 3, 1, 2.

EXAMPLE 10 A school wants to award its students for the value of Honesty, Regularity and Hardwork with a total cash award of ₹6000. Three times the award money for Hardwork added to that given for Honesty amounts to ₹11000. The award money given for Honesty and Hardwork together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from the given three values, suggest one more value which the school must include for awards. [CBSE 2013]

SOLUTION Let the amount of award for Honesty, Regularity and Hardwork be ₹ x , ₹ y and ₹ z respectively. Then,

$$x + y + z = 6000, \quad \dots (i)$$

$$x + 0y + 3z = 11000 \quad \dots (ii)$$

$$\text{and } x + z = 2y \Rightarrow x - 2y + z = 0. \quad \dots (iii)$$

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}.$$

Then, the matrix equation is $AX = B$.

$$\therefore X = A^{-1}B.$$

$$\begin{aligned} \text{Now, } |A| &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -3 & 0 \end{vmatrix} \\ &\quad \{C_2 \rightarrow (C_2 - C_1) \text{ and } C_3 \rightarrow (C_3 - C_1)\} \\ &= 1 \cdot \begin{vmatrix} -1 & 2 \\ -3 & 0 \end{vmatrix} = (0 + 6) = 6 \neq 0. \end{aligned}$$

$\therefore A^{-1}$ exists.

Now, the cofactors of the elements of $|A|$ are given by

$$C_{11} = \begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix} = 6, \quad C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 2, \quad C_{13} = \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} = -2;$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -3, \quad C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \quad C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3;$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3, \quad C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2, \quad C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

$$\therefore (\text{adj } A) = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 3 \\ 3 & -2 & -1 \end{bmatrix}^t = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A) = \frac{1}{6} \cdot \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow X = A^{-1}B = \frac{1}{6} \cdot \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{6} \cdot \begin{bmatrix} 36000 - 33000 + 0 \\ 12000 + 0 + 0 \\ -12000 + 33000 - 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

$$\Rightarrow x = 500, y = 2000 \text{ and } z = 3500.$$

Hence, the award money for Honesty, Regularity and Hardwork is ₹ 500, ₹ 2000 and ₹ 3500 respectively.

Apart from honesty, regularity and hardwork, the school must include an award for a student to be well-behaved.

EXERCISE 8A

Show that each one of the following systems of equations is inconsistent.

1. $x + 2y = 9;$
 $2x + 4y = 7.$

2. $2x + 3y = 5;$
 $6x + 9y = 10.$

3. $4x - 2y = 3;$
 $6x - 3y = 5.$

4. $6x + 4y = 5;$
 $9x + 6y = 8.$

5. $x + y - 2z = 5;$
 $x - 2y + z = -2;$
 $-2x + y + z = 4$

6. $2x - y + 3z = 1;$
 $3x - 2y + 5z = -4;$
 $5x - 4y + 9z = 14.$

7. $x + 2y + 4z = 12;$
 $y + 2z = -1;$
 $3x + 2y + 4z = 4.$

8. $3x - y - 2z = 2;$
 $2y - z = -1;$
 $3x - 5y = 3.$

Solve each of the following systems of equations using matrix method.

9. $5x + 2y = 4;$
 $7x + 3y = 5.$

10. $3x + 4y - 5 = 0;$
 $x - y + 3 = 0.$

11. $x + 2y = 1;$
 $3x + y = 4.$

12. $5x + 7y + 2 = 0;$
 $4x + 6y + 3 = 0.$

13. $2x - 3y + 1 = 0;$
 $x + 4y + 3 = 0.$

14. $4x - 3y = 3;$
 $3x - 5y = 7.$

15. $2x + 8y + 5z = 5;$
 $x + y + z = -2;$
 $x + 2y - z = 2$ [CBSE 2009C]

16. $x - y + z = 1;$
 $2x + y - z = 2;$
 $x - 2y - z = 4$ [CBSE 2006C]

17. $3x + 4y + 7z = 4;$
 $2x - y + 3z = -3;$
 $x + 2y - 3z = 8.$ [CBSE 2012]
18. $x + 2y + z = 7;$
 $x + 3z = 11;$
 $2x - 3y = 1.$ [CBSE 2005, '08, '11]
19. $2x - 3y + 5z = 16;$
 $3x + 2y - 4z = -4;$
 $x + y - 2z = -3.$ [CBSE 2005C]
20. $x + y + z = 4;$
 $2x - y + z = -1;$
 $2x + y - 3z = -9.$ [CBSE 2005]
21. $2x - 3y + 5z = 11;$
 $3x + 2y - 4z = -5;$
 $x + y - 2z = -3.$ [CBSE 2009]
22. $x + y + z = 1;$
 $x - 2y + 3z = 2;$
 $5x - 3y + z = 3.$ [CBSE 2004, '09C]
23. $x + y + z = 6;$
 $x + 2z = 7;$
 $3x + y + z = 12.$ [CBSE 2009]
24. $2x + 3y + 3z = 5;$
 $x - 2y + z = -4;$
 $3x - y - 2z = 3.$ [CBSE 2008C, '12]
25. $4x - 5y - 11z = 12;$
 $x - 3y + z = 1;$
 $2x + 3y - 7z = 2.$ [CBSE 2007]
26. $x - y + 2z = 7;$
 $3x + 4y - 5z = -5;$
 $2x - y + 3z = 12.$ [CBSE 2012]
27. $6x - 9y - 20z = -4;$
 $4x - 15y + 10z = -1;$
 $2x - 3y - 5z = -1.$
28. $3x - 4y + 2z = -1;$
 $2x + 3y + 5z = 7;$
 $x + z = 2.$ [CBSE 2011C]
29. $x + y - z = 1;$
 $3x + y - 2z = 3;$
 $x - y - z = -1.$ [CBSE 2004]
30. $2x + y - z = 1;$
 $x - y + z = 2;$
 $3x + y - 2z = -1.$ [CBSE 2004C]
31. $x + 2y + z = 4;$
 $-x + y + z = 0;$
 $x - 3y + z = 4.$ [CBSE 2012C]
32. $x - y - 2z = 3;$
 $x + y = 1;$
 $x + z = -6.$
33. $5x - y = -7;$
 $2x + 3z = 1;$
 $3y - z = 5.$
34. $x - 2y + z = 0;$
 $y - z = 2;$
 $2x - 3z = 10.$
35. $x - y = 3;$
 $2x + 3y + 4z = 17;$
 $y + 2z = 7.$ [CBSE 2003C, '07C]
36. $4x + 3y + 2z = 60;$
 $x + 2y + 3z = 45;$
 $6x + 2y + 3z = 70.$ [CBSE 2011]

37. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1} . [CBSE 2007C, '08C]

Using A^{-1} , solve the following system of equations:

$$\begin{aligned} 2x - 3y + 5z &= 11; \\ 3x + 2y - 4z &= -5; \\ x + y - 2z &= -3. \end{aligned}$$

38. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$, find A^{-1} .

Using A^{-1} , solve the following system of linear equations:

$$2x + y + z = 1;$$

$$x - 2y - z = \frac{3}{2};$$

$$3y - 5z = 9.$$

HINT: Here $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$.

39. If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$, find AB .

Hence, solve the system of equations:

$$x - 2y = 10, 2x + y + 3z = 8 \text{ and } -2y + z = 7. \quad [\text{CBSE 2011}]$$

HINT: $AB = (11)I \Rightarrow A \left(\frac{1}{11} B \right) = I \Rightarrow A^{-1} = \left(\frac{1}{11} \right) B$.

Using matrices, solve the following system of equations.

40. $\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10, \frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13$ [CBSE 2007C]

41. $\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 4; \frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 0; \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ ($x, y, z \neq 0$)

VALUE BASED QUESTIONS

42. The sum of three numbers is 2. If twice the second number is added to the sum of first and third, we get 1. On adding the sum of second and third numbers to five times the first, we get 6. Find the three numbers by using matrices.
43. The cost of 4 kg potato, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 1 kg potato, 2 kg wheat and 3 kg rice is ₹ 45. The cost of 6 kg potato, 2 kg wheat and 3 kg rice is ₹ 70. Find the cost of each item per kg by matrix method.
44. An amount of ₹ 5000 is put into three investments at 6%, 7% and 8% per annum respectively. The total annual income from these investments is ₹ 358. If the total annual income from first two investments is ₹ 70 more than the income from the third, find the amount of each investment by the matrix method.

HINT: Let these investments be ₹ x , ₹ y and ₹ z respectively.

$$\text{Then, } x + y + z = 5000, \quad \dots (i)$$

$$\frac{6x}{100} + \frac{7y}{100} + \frac{8z}{100} = 358 \Rightarrow 6x + 7y + 8z = 35800 \quad \dots (ii)$$

$$\text{and, } \frac{6x}{100} + \frac{7y}{100} = \frac{8z}{100} + 70 \Rightarrow 6x + 7y - 8z = 7000. \quad \dots (iii)$$

45. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 1,600. School B wants to spend ₹ 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award. [CBSE 2014]

HINT: By the given data, we have

$$\left. \begin{array}{l} 3x + 2y + z = 1600 \\ 4x + y + 3z = 2300 \\ x + y + z = 900 \end{array} \right\}$$

ANSWERS (EXERCISE 8A)

9. $x = 2, y = -3$ 10. $x = -1, y = 2$ 11. $x = \frac{7}{5}, y = \frac{-1}{5}$
 12. $x = \frac{9}{2}, y = \frac{-7}{2}$ 13. $x = \frac{-13}{11}, y = \frac{-5}{11}$ 14. $x = \frac{-6}{11}, y = \frac{-19}{11}$
 15. $x = -3, y = 2, z = -1$ 16. $x = 1, y = -1, z = -1$ 17. $x = 1, y = 2, z = -1$
 18. $x = 2, y = 1, z = 3$ 19. $x = 2, y = 1, z = 3$ 20. $x = 2, y = 1, z = 3$
 21. $x = 1, y = 2, z = 3$ 22. $x = \frac{1}{2}, y = 0, z = \frac{1}{2}$ 23. $x = 3, y = 1, z = 2$
 24. $x = 1, y = 2, z = -1$ 25. $x = -1, y = -1, z = -1$ 26. $x = 1, y = 2, z = 5$
 27. $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{5}$ 28. $x = 3, y = 2, z = -1$ 29. $x = 2, y = 1, z = 2$
 30. $x = 1, y = 2, z = 3$ 31. $x = 2, y = 0, z = 2$ 32. $x = -2, y = 3, z = -4$
 33. $x = -1, y = 2, z = 1$ 34. $x = 2, y = 0, z = -2$ 35. $x = 2, y = -1, z = 4$
 36. $x = 5, y = 8, z = 8$ 37. $x = 1, y = 2, z = 3$ 38. $x = 1, y = \frac{1}{2}, z = \frac{-3}{2}$
 39. $x = 4, y = -3, z = 1$ 40. $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{5}$ 41. $x = \frac{1}{2}, y = -1, z = 1$
 42. 1, -1, 2 43. ₹ 5, ₹ 8, ₹ 8 44. ₹ 1000, ₹ 2200, ₹ 1800
 45. ₹ 200 for sincerity, ₹ 300 for truthfulness and ₹ 400 for helpfulness.

One more value may be like honesty, kindness, etc.

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. If A and B are 2-rowed square matrices such that

$$(A + B) = \begin{bmatrix} 4 & -3 \\ 1 & 6 \end{bmatrix} \text{ and } (A - B) = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix} \text{ then } AB = ?$$

(a) $\begin{bmatrix} -7 & 5 \\ 1 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -5 \\ 1 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & -1 \\ 5 & -5 \end{bmatrix}$ (d) $\begin{bmatrix} 7 & -1 \\ -5 & 5 \end{bmatrix}$

2. If $\begin{bmatrix} 3 & -2 \\ 5 & 6 \end{bmatrix} + 2A = \begin{bmatrix} 5 & 6 \\ -7 & 10 \end{bmatrix}$ then $A = ?$

(a) $\begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 5 \\ -3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 \\ -6 & 2 \end{bmatrix}$ (d) none of these

3. If $A = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ -6 & 2 \end{bmatrix}$ are such that $4A + 3X = 5B$ then $X = ?$

(a) $\begin{bmatrix} 4 & -5 \\ -6 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 5 \\ -6 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} -4 & 5 \\ 6 & -2 \end{bmatrix}$ (d) none of these

4. If $(A - 2B) = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$ and $(2A - 3B) = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$ then $B = ?$

(a) $\begin{bmatrix} 6 & -4 \\ -3 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & 6 \\ -3 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & -6 \\ 3 & -3 \end{bmatrix}$ (d) none of these

5. If $(2A - B) = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$ and $(2B + A) = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix}$ then $A = ?$

(a) $\begin{bmatrix} -3 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$ (d) none of these

6. If $2 \begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$ then

(a) $(x = -2, y = 8)$ (b) $(x = 2, y = -8)$

(c) $(x = 3, y = -6)$ (d) $(x = -3, y = 6)$

7. If $\begin{bmatrix} x - y & 2x - y \\ 2x + z & 3z + w \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 5 & 13 \end{bmatrix}$ then

(a) $z = 3, w = 4$ (b) $z = 4, w = 3$

(c) $z = 1, w = 2$ (d) $z = 2, w = -1$

8. If $\begin{bmatrix} x & y \\ 3y & x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ then
 (a) $x = 1, y = 2$ (b) $x = 2, y = 1$ (c) $x = 1, y = 1$ (d) none of these
9. If the matrix $A = \begin{bmatrix} 3 - 2x & x + 1 \\ 2 & 4 \end{bmatrix}$ is singular then $x = ?$
 (a) 0 (b) 1 (c) -1 (d) -2
10. If $A_\alpha = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ then $(A_\alpha)^2 = ?$
 (a) $\begin{bmatrix} \cos^2 \alpha & \sin^2 \alpha \\ -\sin^2 \alpha & \cos^2 \alpha \end{bmatrix}$ (b) $\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$
 (c) $\begin{bmatrix} 2\cos \alpha & 2\sin \alpha \\ -\sin \alpha & 2\cos \alpha \end{bmatrix}$ (d) none of these
11. If $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ be such that $A + A' = I$, then $\alpha = ?$
 (a) π (b) $\frac{\pi}{3}$ (c) π (d) $\frac{2\pi}{3}$
12. If $A = \begin{bmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{bmatrix}$ is singular then $k = ?$
 (a) $\frac{16}{3}$ (b) $\frac{34}{5}$ (c) $\frac{33}{2}$ (d) none of these
13. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\text{adj } A = ?$
 (a) $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ (b) $\begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$ (c) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (d) $\begin{bmatrix} -d & -b \\ c & a \end{bmatrix}$
14. If $A = \begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ then $x = ?$
 (a) 1 (b) 2 (c) $\frac{1}{2}$ (d) -2
15. If A and B are square matrices of the same order then $(A + B)(A - B) = ?$
 (a) $(A^2 - B^2)$ (b) $A^2 + AB - BA - B^2$
 (c) $A^2 - AB + BA - B^2$ (d) none of these
16. If A and B are square matrices of the same order then $(A + B)^2 = ?$
 (a) $A^2 + 2AB + B^2$ (b) $A^2 + AB + BA + B^2$
 (c) $A^2 + 2BA + B^2$ (d) none of these

17. If A and B are square matrices of the same order then $(A - B)^2 = ?$
 (a) $A^2 - 2AB + B^2$ (b) $A^2 - AB - BA + B^2$
 (c) $A^2 - 2BA + B^2$ (d) none of these
18. If A and B are symmetric matrices of the same order then $(AB - BA)$ is always
 (a) a symmetric matrix (b) a skew-symmetric matrix
 (c) a zero matrix (d) an identity matrix
19. Matrices A and B are inverses of each other only when
 (a) $AB = BA$ (b) $AB = BA = O$ (c) $AB = O, BA = I$ (d) $AB = BA = I$
20. For square matrices A and B of the same order, we have $\text{adj}(AB) = ?$
 (a) $(\text{adj } A)(\text{adj } B)$ (b) $(\text{adj } B)(\text{adj } A)$
 (c) $|AB|$ (d) none of these
21. If A is a 3-rowed square matrix and $|A| = 4$ then $\text{adj}(\text{adj } A) = ?$
 (a) $4A$ (b) $16A$ (c) $64A$ (d) none of these
22. If A is a 3-rowed square matrix and $|A| = 5$ then $|\text{adj } A| = ?$
 (a) 5 (b) 25 (c) 125 (d) none of these
23. For any two matrices A and B ,
 (a) $AB = BA$ is always true
 (b) $AB = BA$ is never true
 (c) sometimes $AB = BA$ and sometimes $AB \neq BA$
 (d) whenever AB exists, then BA exists
24. If $A \cdot \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ then $A = ?$
 (a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (d) none of these
25. If A is an invertible square matrix then $|A^{-1}| = ?$
 (a) $|A|$ (b) $\frac{1}{|A|}$ (c) 1 (d) 0
26. If A and B are invertible matrices of the same order then $(AB)^{-1} = ?$
 (a) $(A^{-1} \times B^{-1})$ (b) $(A \times B^{-1})$ (c) $(A^{-1} \times B)$ (d) $(B^{-1} \times A^{-1})$
27. If A and B are two nonzero square matrices of the same order such that $AB = 0$ then
 (a) $|A| = 0$ or $|B| = 0$ (b) $|A| = 0$ and $|B| = 0$
 (c) $|A| \neq 0$ and $|B| \neq 0$ (d) none of these
28. If A is a square matrix such that $|A| \neq 0$ and $A^2 - A + 2I = O$ then $A^{-1} = ?$
 (a) $(I - A)$ (b) $(I + A)$ (c) $\frac{1}{2}(I - A)$ (d) $\frac{1}{2}(I + A)$

29. If $A = \begin{bmatrix} 1 & \lambda & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ is not invertible then $\lambda = ?$
- (a) 2 (b) 1 (c) -1 (d) 0
30. If $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ then $A^{-1} = ?$
- (a) A (b) $-A$ (c) $\text{adj } A$ (d) $-\text{adj } A$
31. The matrix $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is
- (a) idempotent (b) orthogonal
(c) nilpotent (d) none of these
32. The matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is
- (a) nonsingular (b) idempotent
(c) nilpotent (d) orthogonal
33. If A is singular then $A(\text{adj } A) = ?$
- (a) a unit matrix (b) a null matrix
(c) a symmetric matrix (d) none of these
34. For any 2-rowed square matrix A , if $A \cdot (\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$ then the value of $|A|$ is
- (a) 0 (b) 8 (c) 64 (d) 4
35. If $A = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}$ then $|A^{-1}| = ?$
- (a) -5 (b) $\frac{-1}{5}$ (c) $\frac{1}{25}$ (d) 25
36. If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$ and $A^2 + xI = yA$ then the values of x and y are
- (a) $x = 6, y = 6$ (b) $x = 8, y = 8$ (c) $x = 5, y = 8$ (d) $x = 6, y = 8$
37. If matrices A and B anticommute then
- (a) $AB = BA$ (b) $AB = -BA$ (c) $(AB) = (BA)^{-1}$ (d) none of these
38. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ then $\text{adj } A = ?$
- (a) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$ (d) none of these

39. If $A = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$ and B is a square matrix of order 2 such that $AB = I$ then $B = ?$

- (a) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ (d) none of these

40. If A and B are invertible square matrices of the same order then $(AB)^{-1} = ?$

- (a) AB^{-1} (b) $A^{-1}B$ (c) $A^{-1}B^{-1}$ (d) $B^{-1}A^{-1}$

41. If $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, then $A^{-1} = ?$

- (a) $\begin{bmatrix} \frac{3}{7} & \frac{-1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{3} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$ (d) none of these

42. If $|A| = 3$ and $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \\ \frac{-5}{3} & \frac{2}{3} \end{bmatrix}$ then $\text{adj } A = ?$

- (a) $\begin{bmatrix} 9 & 3 \\ -5 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}$
 (c) $\begin{bmatrix} -9 & 3 \\ 5 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 9 & -3 \\ 5 & -2 \end{bmatrix}$

43. If A is an invertible matrix and $A^{-1} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ then $A = ?$

- (a) $\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} \end{bmatrix}$ (c) $\begin{bmatrix} -3 & 2 \\ \frac{5}{2} & \frac{-3}{2} \end{bmatrix}$ (d) none of these

44. If $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ and $f(x) = 2x^2 - 4x + 5$ then $f(A) = ?$

- (a) $\begin{bmatrix} 19 & -32 \\ -16 & 51 \end{bmatrix}$ (b) $\begin{bmatrix} 19 & -16 \\ -32 & 51 \end{bmatrix}$
 (c) $\begin{bmatrix} 19 & -11 \\ -27 & 51 \end{bmatrix}$ (d) none of these

45. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ then $A^2 - 4A = ?$

- (a) I (b) $5I$ (c) $3I$ (d) 0

46. If A is a 2-rowed square matrix and $|A| = 6$ then $A \cdot \text{adj } A = ?$

(a) $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$

(d) none of these

47. If A is an invertible square matrix and k is a non-negative real number then $(kA)^{-1} = ?$

(a) $k \cdot A^{-1}$

(b) $\frac{1}{k} \cdot A^{-1}$

(c) $-k \cdot A^{-1}$

(d) none of these

48. If $A = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 0 & -2 \\ -2 & -1 & 2 \end{bmatrix}$ then $A^{-1} = ?$

(a) $\begin{bmatrix} 2 & 9 & -8 \\ -2 & 8 & 7 \\ -1 & 5 & -4 \end{bmatrix}$

(b) $\begin{bmatrix} -2 & 9 & -8 \\ 2 & 8 & 7 \\ -1 & -5 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & -9 & -8 \\ 2 & 8 & 7 \\ -1 & -5 & -4 \end{bmatrix}$

(d) none of these

49. If A is a square matrix then $(A + A')$ is

(a) a null matrix

(b) an identity matrix

(c) a symmetric matrix

(d) a skew-symmetric matrix

50. If A is a square matrix then $(A - A')$ is

(a) a null matrix

(b) an identity matrix

(c) a symmetric matrix

(d) a skew-symmetric matrix

51. If A is a 3-rowed square matrix and $|3A| = k|A|$ then $k = ?$

(a) 3

(b) 9

(c) 27

(d) 1

52. Which one of the following is a scalar matrix?

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix}$

(d) none of these

53. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = (A^2 + B^2)$ then

(a) $a = 2, b = -3$

(b) $a = -2, b = 3$

(c) $a = 1, b = 4$

(d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (b) 2. (c) 3. (a) 4. (b) 5. (c) 6. (b) 7. (a) 8. (c) 9. (b) 10. (b)
 11. (b) 12. (c) 13. (c) 14. (c) 15. (c) 16. (b) 17. (b) 18. (b) 19. (d) 20. (b)
 21. (a) 22. (b) 23. (c) 24. (c) 25. (b) 26. (d) 27. (b) 28. (c) 29. (b) 30. (c)
 31. (c) 32. (b) 33. (b) 34. (b) 35. (b) 36. (b) 37. (b) 38. (a) 39. (c) 40. (d)
 41. (b) 42. (b) 43. (c) 44. (b) 45. (b) 46. (a) 47. (b) 48. (c) 49. (c) 50. (d)
 51. (c) 52. (c) 53. (c)

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

1. $2A = (A + B) + (A - B)$ and $2B = (A + B) - (A - B)$.
 2. $2A = \begin{bmatrix} 5 & 6 \\ -7 & 10 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -12 & 4 \end{bmatrix}$.
 3. $4A + 3X = 5B \Rightarrow 3X = (5B - 4A) \Rightarrow X = \frac{1}{3}(5B - 4A)$.
 4. $B = (2A - 3B) - 2(A - 2B)$.
 5. $5A = 2(2A - B) + (2B + A)$. Then, $A = \frac{1}{5}(5A)$.
 6. $[2x + 1 = 5 \Rightarrow x = 2]$ and $[8 + y = 0 \Rightarrow y = -8]$.
 7. $(x - y = -1 \text{ and } 2x - y = 0) \Rightarrow (x = 1, y = 2)$
 $(2x + z = 5 \Rightarrow z = 3)$ and $(3z + w = 13 \Rightarrow w = 4)$.
 8. Solve $x + 2y = 3$ and $3y + 2x = 5$.
 9. A is singular $\Leftrightarrow |A| = 0$.
 12. A is singular $\Leftrightarrow |A| = 0$.
 14. Use $AA^{-1} = I$.
 15. Using distributive law, we have
 $(A + B) \cdot (A - B) = A(A - B) + B(A - B) = (A^2 - AB + BA - B^2)$.
 16. $(A + B)^2 = (A + B) \cdot (A + B) = A(A + B) + B(A + B) = (A^2 + AB + BA + B^2)$.
 17. $(A - B)^2 = (A - B) \cdot (A - B) = A(A - B) - B(A - B) = (A^2 - AB - BA + B^2)$.
 18. Given $A' = A$ and $B' = B$.
 $\therefore (AB - BA)' = (AB)' - (BA)' = (B'A' - A'B') = (BA - AB) = -(AB - BA)$
 $\therefore (AB - BA)$ is skew symmetric.
 19. A and B are inverses of each other only when $AB = BA = I$.
 20. $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$.
 21. $\text{adj}(\text{adj } A) = |A|^{(n-2)} = |A|^{(3-2)} \cdot A = |A| \cdot A = 4A$.
 22. $|\text{adj } A| = |A|^{(n-1)} = |A|^2 = 5^2 = 25$.
 24. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. Find a, b, c and d .

$$25. AA^{-1} = I \Rightarrow |AA^{-1}| = |I| = 1$$

$$\Rightarrow |A| \cdot |A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}.$$

$$26. (AB)^{-1} = B^{-1}A^{-1}.$$

$$27. [AB = 0 \text{ and } A \neq 0, B \neq 0] \Rightarrow |A| = 0 \text{ and } |B| = 0.$$

$$28. 2I = (A - A^2) \Rightarrow 2A^{-1} = A^{-1}A - A^{-1}AA = I - IA = (I - A)$$

$$\therefore \Rightarrow A^{-1} = \frac{1}{2}(I - A).$$

$$29. A \text{ is not invertible} \Leftrightarrow |A| = 0.$$

$$30. |A| = 1 \Rightarrow A^{-1} = \frac{1}{|A|} \text{adj } A = (\text{adj } A).$$

$$31. A^2 = O \Rightarrow A \text{ is nilpotent.}$$

$$32. A^2 = A \Rightarrow A \text{ is idempotent.}$$

$$33. \text{ Given } |A| = 0. \text{ So, } A(\text{adj } A) = |A| \cdot I = 0 \cdot I = 0.$$

$$\therefore A(\text{adj } A) \text{ is a null matrix.}$$

$$34. A \cdot \text{adj } A = |A| \cdot I = 8 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 8I \rightarrow |A| = 8.$$

$$35. AA^{-1} = I \Rightarrow |AA^{-1}| = |I| \Rightarrow |A| \cdot |A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}.$$

$$|A| = \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = (-2 - 3) = (-5) \Rightarrow |A^{-1}| = \frac{-1}{5}.$$

$$36. \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} + x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix} \Rightarrow \begin{bmatrix} 16+x & 8 \\ 56 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}.$$

$$\therefore y = 8 \text{ and } (16 + x = 3y = 3 \times 8 = 24) \Rightarrow x = 8.$$

$$37. A \text{ and } B \text{ anticommute} \Leftrightarrow AB = -BA.$$

$$39. AB = I \Rightarrow B = A^{-1}.$$

$$40. (AB)^{-1} = B^{-1}A^{-1}.$$

$$41. |A| = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = (6 + 1) = 7 \neq 0$$

$$M_{11} = 3, M_{12} = 1, M_{21} = -1 \text{ and } M_{22} = 2$$

$$\therefore C_{11} = 3, C_{12} = -1, C_{21} = 1 \text{ and } C_{22} = 2$$

$$\Rightarrow \text{Adj } A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} (\text{adj } A) = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}.$$

$$42. A^{-1} = \frac{1}{|A|} \cdot \text{adj } A \rightarrow \text{adj } A = |A| \cdot A^{-1} = 3A^{-1} = \begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}.$$

$$43. A = (A^{-1})^{-1}. \text{ So, find the inverse of } A^{-1}.$$

$$44. f(A) = 2A^2 - 4A + 5I.$$

$$45. A^2 - 4A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5I.$$

$$46. A \cdot (\text{adj } A) = |A| \cdot I = 6 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}.$$

$$47. (kA)^{-1} = \frac{1}{k} \cdot A^{-1} \text{ is true.}$$

$$48. A^{-1} = \frac{1}{|A|} \cdot \text{adj } A.$$

$$49. A \text{ is a square matrix} \Rightarrow (A + A') \text{ is symmetric.}$$

$$50. A \text{ is a square matrix} \Rightarrow (A - A') \text{ is skew-symmetric.}$$

$$51. |3A| = (3 \times 3 \times 3)|A| = 27 \cdot |A|.$$

$$52. \text{ A scalar matrix is a square matrix each of whose non-diagonal elements is 0 and all diagonal elements are equal.}$$

$$53. (A + B)^2 = (A^2 + B^2) \Leftrightarrow A^2 + B^2 + AB + BA = (A^2 + B^2) \Leftrightarrow AB = -BA.$$

$$AB = -BA \Leftrightarrow \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} = \begin{bmatrix} -a-2 & a+1 \\ -b+2 & b-1 \end{bmatrix}$$

$$\text{Now, } (a+1=2 \text{ and } b-1=3) \Rightarrow (a=1 \text{ and } b=4).$$
